Universal scaling and the essential singularity at the abrupt Ising transition

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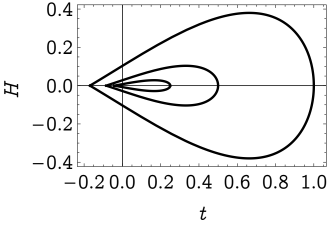
16 March 2016

Parametric Ising Models



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a plot of susceptibility with precision parametric fit and it isn't very good at the abrupt transition

Scaling forms of Ising variables that do well globally.

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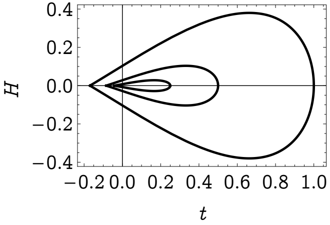

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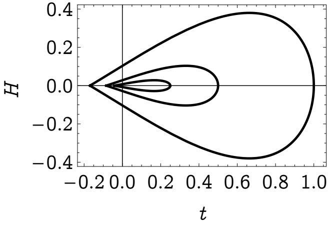

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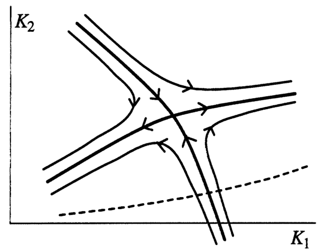
Typically do a very poor job near the abrupt transition at $H = 0$.

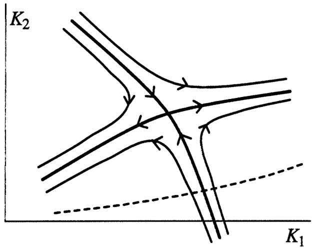
Renormalization and Universality



From *Scaling and Renormalization in Statistical Physics* by John Cardy

Renormalization is an analytic scaling transformation that acts on system space.

Renormalization and Universality



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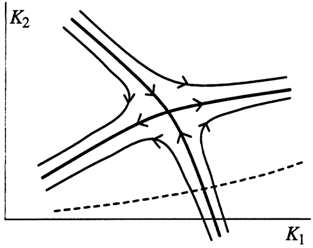

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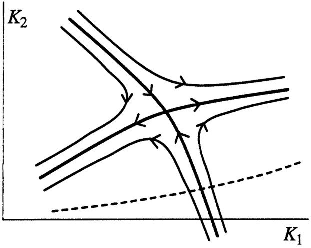

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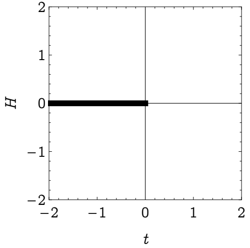
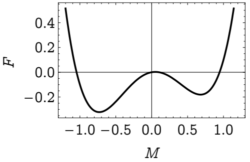
Not all nonanalytic behavior are power laws!

The Metastable Ising Model





Consider an Ising-class model brought into a metastable state.

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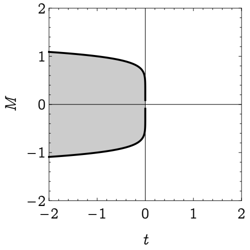

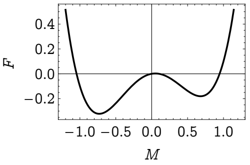

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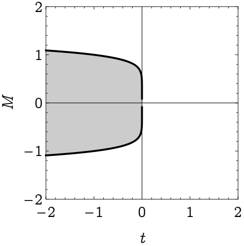

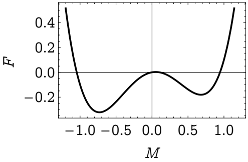

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A domain of N spins entering the stable phase causes a free energy change

$$\Delta F = \Sigma N^{\sigma} - MHN$$

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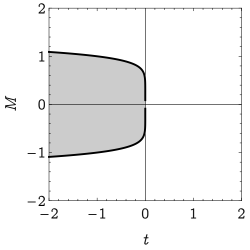

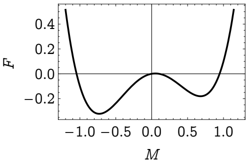

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The metastable phase is stable to domains smaller than

$$N_{\text{crit}} = \left(\frac{MH}{\sigma\Sigma}\right)^{-1/(\sigma-1)}$$

but those larger will grow to occupy the entire system.

The Metastable Ising Model

The formation of a critical domain has energy cost

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Decay of the equilibrium state implies existence of an imaginary part in the free energy,

$$\operatorname{Im} F \sim e^{-\beta \Delta F_{\text{crit}}}$$

The Metastable Ising Model

Near the Ising critical point, $\sigma = 1 - \frac{1}{d}$ and

$$M = t^{\beta} \mathcal{M}(h/t^{\beta\delta}) \qquad \Sigma = t^{\mu} \mathcal{S}(h/t^{\beta\delta})$$

with $\mathcal{M}(0)$ and $\mathcal{S}(0)$ nonzero and finite.

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with $\mathcal{M}(0)$ and $\mathcal{S}(0)$ nonzero and finite.

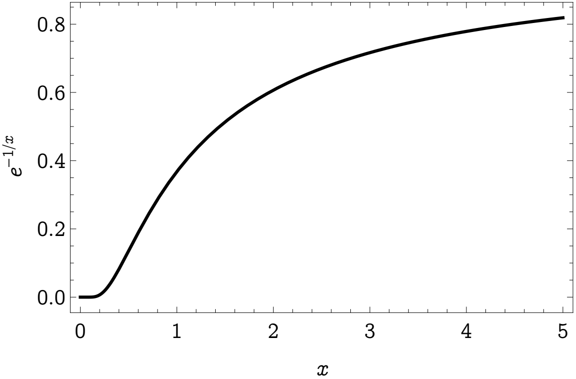
Therefore,

$$\Delta F_{\text{crit}} \sim \Sigma\left(\frac{MH}{\Sigma}\right)^{-(d-1)} = X^{-(d-1)}\mathcal{F}(X)$$

for $X = h/t^{\beta\delta}$, and

$$\operatorname{Im} F = t^{2-\alpha}\mathcal{I}(X)e^{-\beta/X^{(d-1)}}$$

The Essential Singularity



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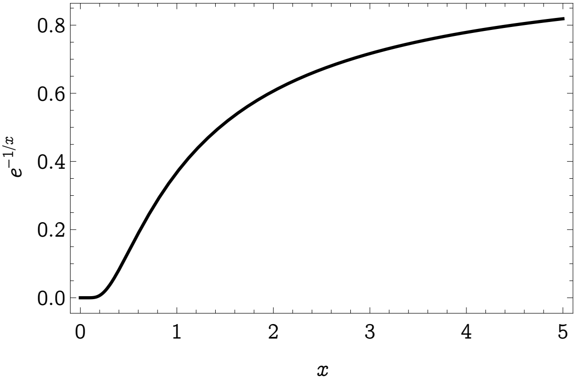

Imaginary free energy is nonanalytic at $H = 0$.

This and its implications are therefore a universal feature of the Ising class.

The Essential Singularity

Analytic properties of the partition function imply that

$$F(X) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X' - X} \, \mathrm{d}X'$$

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Only predictive for high moments of F, or

$$f_n = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X'^{n+1}} \, \mathrm{d}X'$$

for $F = \sum f_n X^n$.

The Essential Singularity

Results from field theory indicate that $\mathcal{I}(X) \propto X + \mathcal{O}(X^2)$ for $d = 2$, so that

$$\operatorname{Im} F = t^{2-\alpha}\left(AX + \mathcal{O}(X^2)\right) e^{-\beta/X^{(d-1)}}$$

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Not a convergent series—the real part of F for $H > 0$ is also nonanalytic!

The Essential Singularity

In two dimensions, the Cauchy integral does not converge, normalize with λ,

$$F(X \mid \lambda) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X' - X} \frac{1}{1 + (\lambda X')^2} \, \mathrm{d}X'$$

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Exact result has form

$$F(X \mid \lambda) = \frac{A}{\pi} \frac{1}{1 + (\lambda X)^2} \Big[X e^{B/X} \operatorname{Ei}(-B/X) + \frac{1}{\lambda} \operatorname{Im}\big(e^{-i\lambda B}(i + \lambda X)(\pi + i \operatorname{Ei}(i\lambda B))\big) \Big]$$

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The Cauchy integral is only predictive for high moments.

The Essential Singularity

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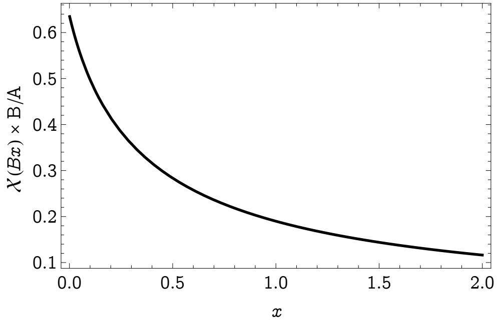
Has a well-defined limit as $\lambda \to 0$, simple functional form:

$$\chi = t^{-\gamma}\mathcal{X}(h/t^{\beta\delta})$$

where the scaling function is

$$\mathcal{X}(X) = \frac{A}{\pi X^3}\left[(B - X)X + B^2 e^{B/X}\,\mathrm{Ei}(-B/X)\right]$$



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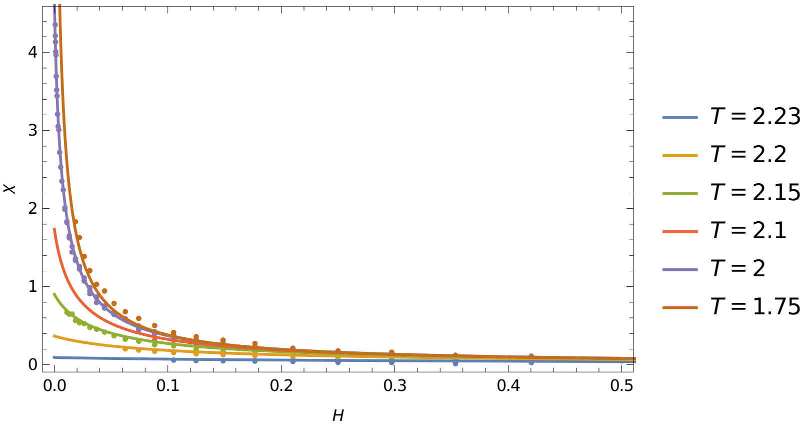
Two parameter fit to simulations yields $A = -0.0939(8)$, $B = 5.45(6)$, close agreement in limit of small t and H!



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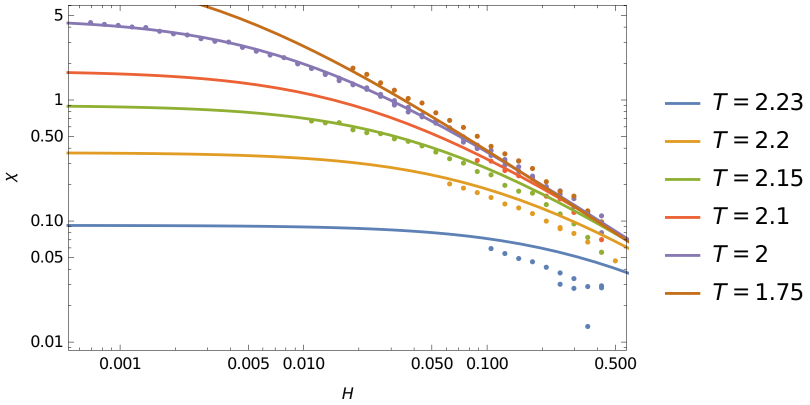

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Remain on the lookout for other universal properties to incorporate.

Questions?