Universal scaling and the essential singularity at the abrupt Ising transition

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Outline

- Renormalization and the Ising model

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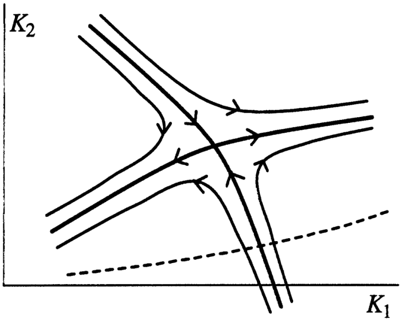
- Closed-form results for the 2D Ising susceptibility

Renormalization and the Ising Model



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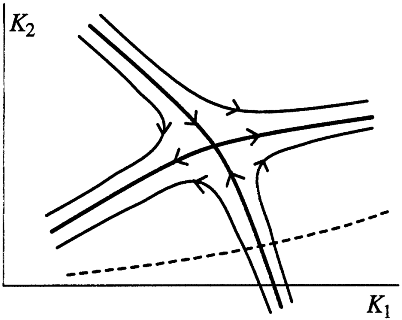

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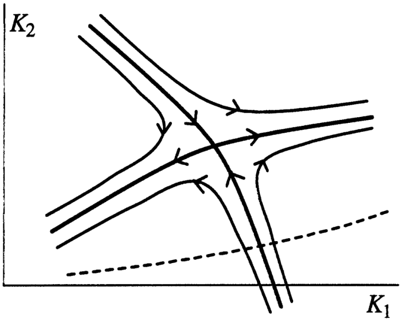

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Nonanalytic behavior is preserved by RG.

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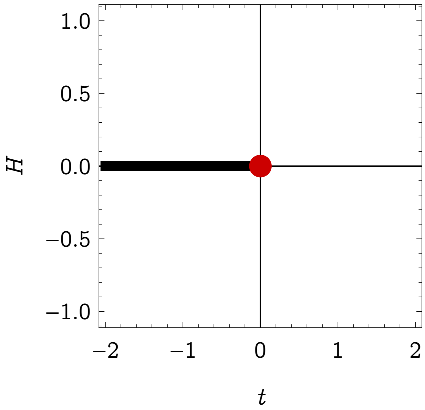

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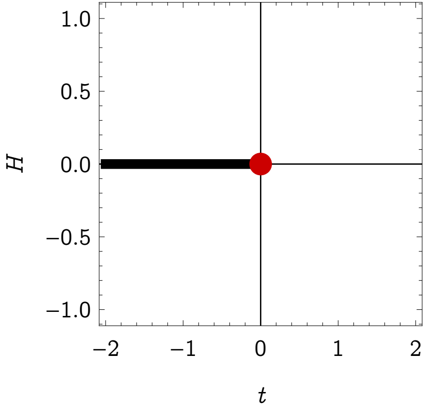

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Connected to line of abrupt transitions.

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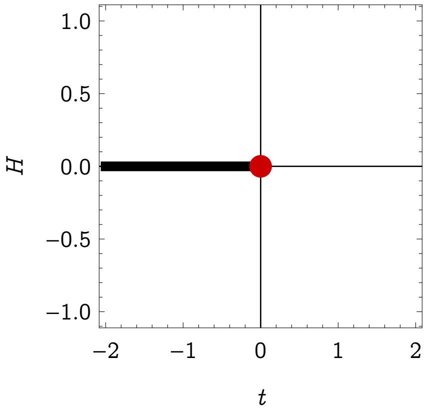

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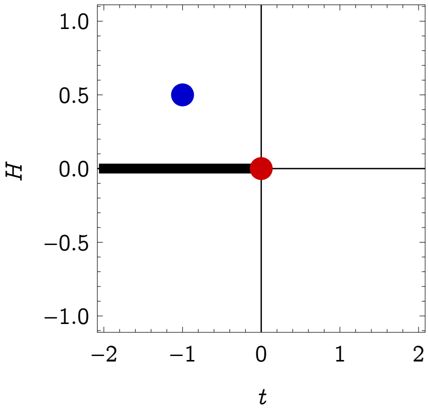
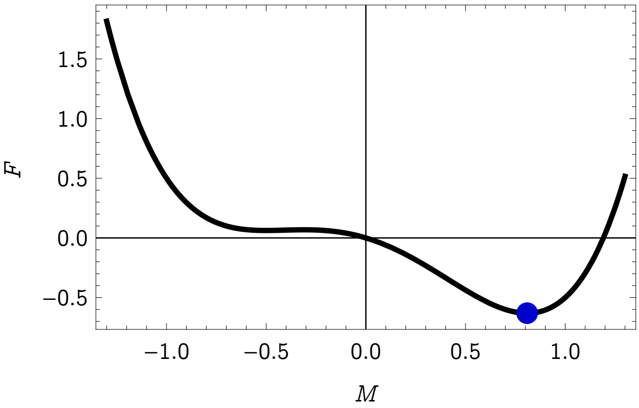
We've identified nonanalytic behavior along the abrupt transition line.

Metastability & Complex Free Energy





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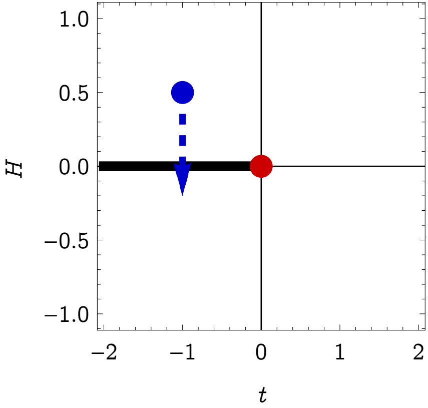

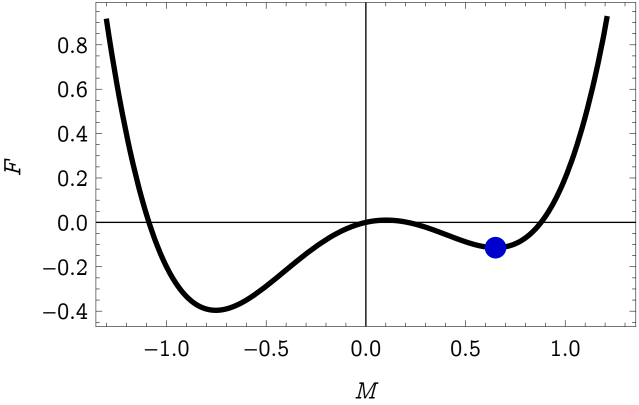

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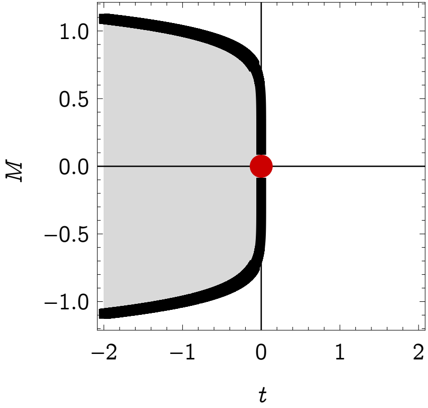
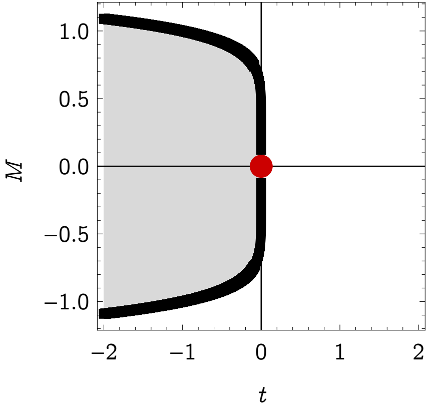


Thermodynamics can be continued into metastable phase.

Metastability & Complex Free Energy



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Decay rate related to imaginary free energy, $\Gamma \propto \frac{1}{kT}\,\mathrm{Im}\,F$ (Langer 1969).

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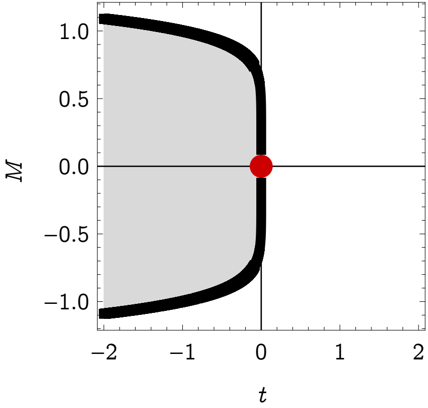

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Ising metastable decay somewhat well studied (Günther 1980, Houghton 1980)

Metastability & Complex Free Energy

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Metastable phase is stable to domains smaller than

$$N_{\text{crit}} = \left(\frac{MH}{\sigma\Sigma}\right)^{-1/(\sigma-1)}$$

but larger will grow to occupy the entire system, decay to stable phase.

Metastability & Complex Free Energy

Formation of critical domain has energy cost

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Imaginary free energy is therefore

$$\operatorname{Im} F \sim \Gamma \sim P_{\text{crit}} \sim e^{-\beta \Delta F_{\text{crit}}} = e^{-\beta(\Sigma/(MH)^{\sigma})^{1/(1-\sigma)}}$$

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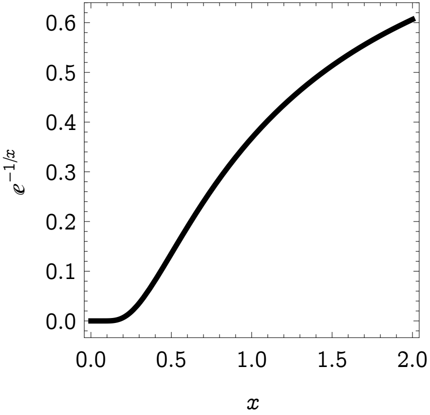

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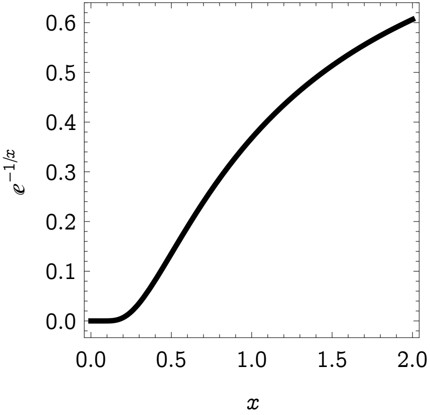

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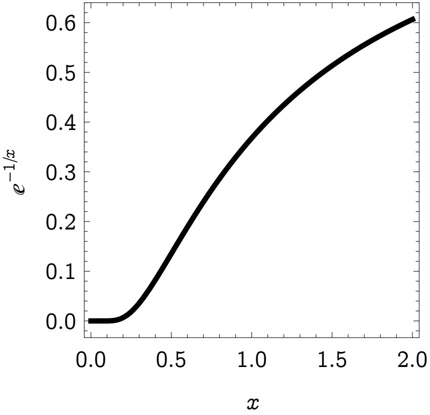

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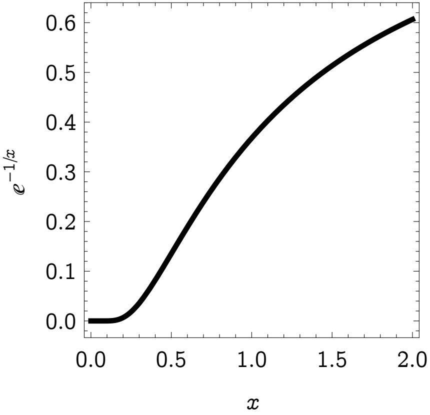

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Nonanalytic behavior is universal!

Can directly observe by measuring metastable decay rate, but what else?

Analytic Constrains on the Stable Free Energy

The Metastable Ising Model

Near the Ising critical point, $\sigma = 1 - \frac{1}{d}$ and

$$M = t^{\beta} \mathcal{M}(h/t^{\beta\delta}) \qquad \Sigma = t^{\mu} \mathcal{S}(h/t^{\beta\delta})$$

with $\mathcal{M}(0)$ and $\mathcal{S}(0)$ nonzero and finite.

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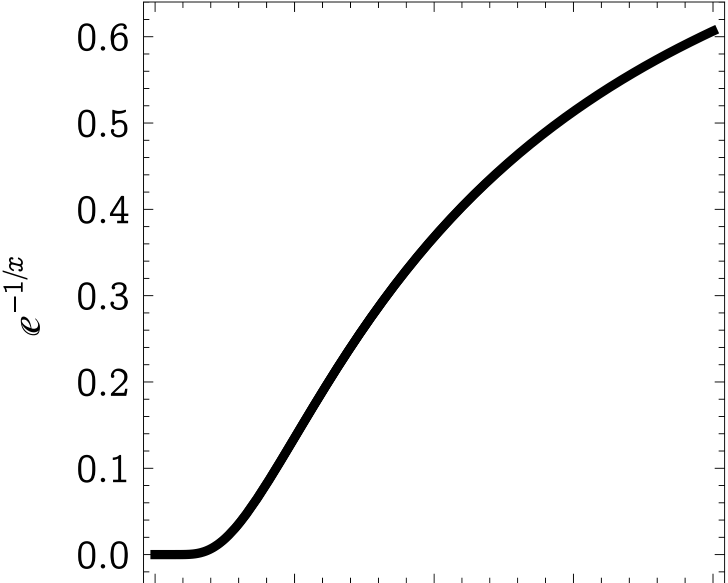
Therefore,

$$\Delta F_{\text{crit}} \sim \Sigma\left(\frac{MH}{\Sigma}\right)^{-(d-1)} = X^{-(d-1)}\mathcal{F}(X)$$

for $X = h/t^{\beta\delta}$, and

$$\operatorname{Im} F = t^{2-\alpha}\mathcal{I}(X)e^{-\beta/X^{(d-1)}}$$

The Essential Singularity



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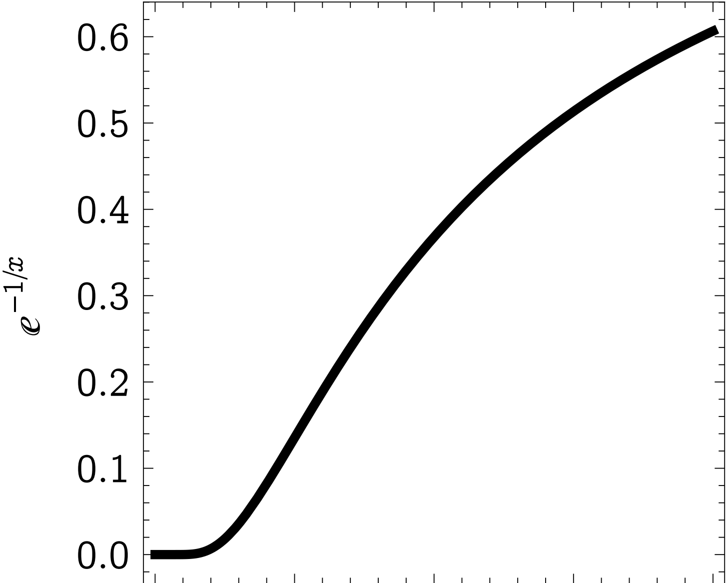

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Analytic properties of the partition function imply that

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Only predictive for high moments of F, or

$$f_n = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X'^{n+1}} \, \mathrm{d}X'$$

for $F = \sum f_n X^n$.

The Essential Singularity

Results from field theory indicate that $\mathcal{I}(X) \propto X + \mathcal{O}(X^2)$ for $d = 2$, so that

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Not a convergent series—the real part of F for $H > 0$ is also nonanalytic!

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In two dimensions, the Cauchy integral does not converge, normalize with λ,

$$F(X \mid \lambda) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X' - X} \frac{1}{1 + (\lambda X')^2} \, \mathrm{d}X'$$

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$$\begin{aligned} F(X \mid \lambda) = \frac{A}{\pi} \frac{1}{1 + (\lambda X)^2} \Big[& X e^{B/X} \operatorname{Ei}(-B/X) \\ & + \frac{1}{\lambda} \operatorname{Im}\left(e^{-i\lambda B}(i + \lambda X)(\pi + i \operatorname{Ei}(i\lambda B))\right) \Big] \end{aligned}$$

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The Cauchy integral is only predictive for high moments.

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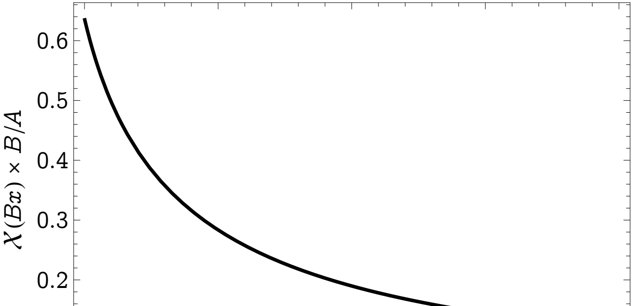
Has a well-defined limit as $\lambda \to 0$, simple functional form:

$$\chi = t^{-\gamma} \mathcal{X}(h/t^{\beta\delta})$$

where the scaling function is

$$\mathcal{X}(X) = \frac{A}{\pi X^3}\left[(B - X)X + B^2 e^{B/X} \operatorname{Ei}(-B/X)\right]$$



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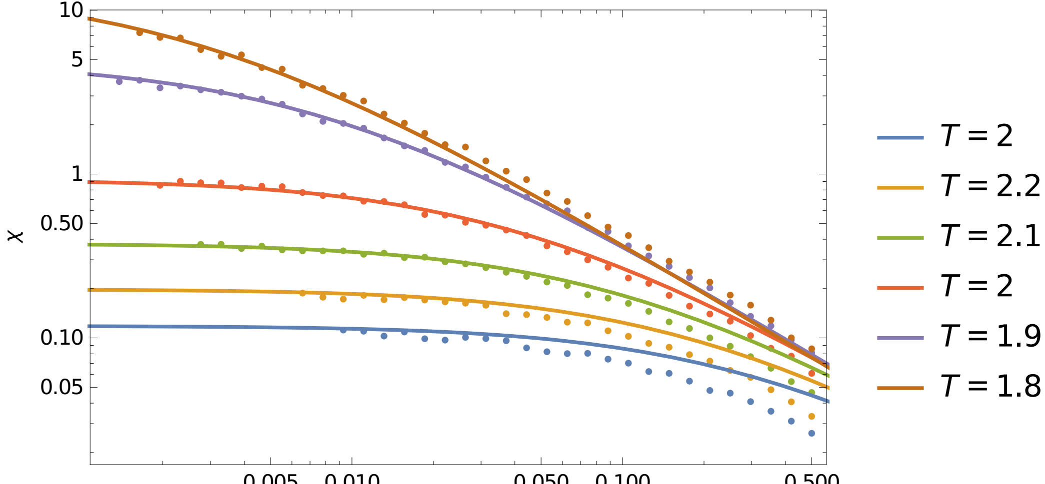

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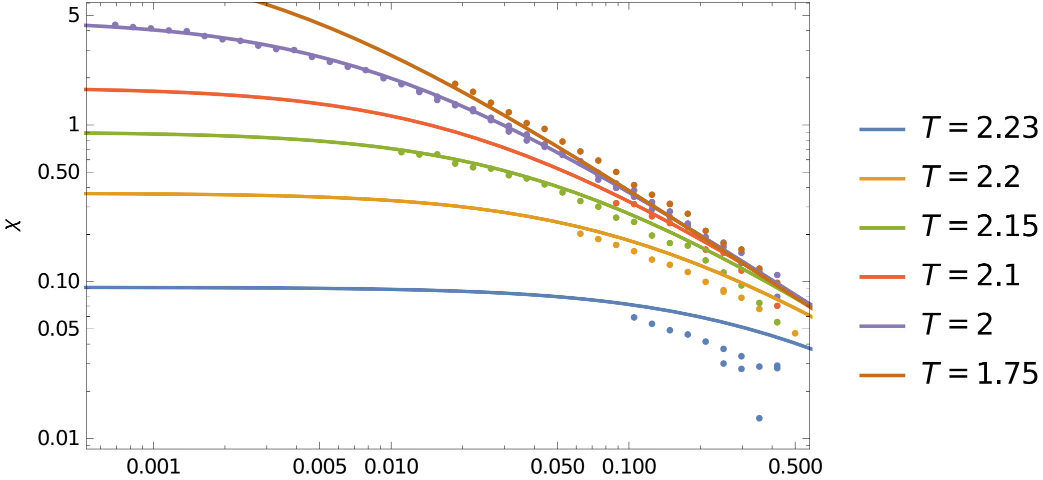

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Remain on the lookout for other universal properties to incorporate.

Questions?