Universal scaling and the essential singularity at the Ising first-order transition

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Renormalization and free energy

Rescale a system by a factor b, with couplings $K \to K'$. From John Cardy's *Scaling and Renormalization in Statistical Physics*, free energy per site f

$$f(\{K\}) = g(\{K\}) + b^{-d} f(\{K'\})$$

> *However, if we are interested in extracting only the singular behavior of f, ... we may obtain a homogeneous transformation law for the singular part of the free energy f_s*
>
> $$f_s(\{K\}) = b^{-d} f_s(\{K'\})$$

Defense: $g(\{K\})$ is an analytic function of $\{K\}$, while the singular part is nonanalytic

Follow thermodynamic functions onto metastable branch.

$$\Delta f \sim \Sigma\gamma(N) - HMN$$

Near the critical point, $\gamma(N) \sim N^{\frac{d-1}{d}}$

$$M = |t|^{\beta}\mathcal{M}(h/t^{\beta\delta})$$

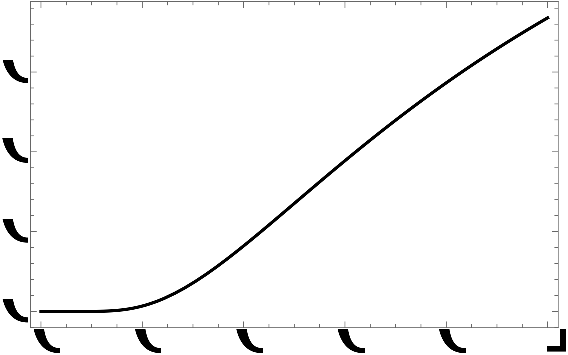
$$N_{crit} \sim \left(\frac{\Sigma}{HM}\left(1 - \tfrac{1}{d}\right)\right)^{d}$$

$$\Delta f_{crit} \sim \Sigma\left(\frac{\Sigma}{HM}\right)^{d-1} \sim X^{-(d-1)}\frac{\mathcal{S}^{d}(X)}{\mathcal{M}^{d-1}(X)}$$

$X = h/t^{\beta\delta}$ The probability that such a domain forms and the metastable state decays is given by the Boltzmann factor, so that $\Sigma \sim |t|^{\mu}$, $\mu = -\nu + \gamma + 2\beta$

$$\mathrm{Im}\, f \sim e^{-\beta\,\Delta f_{crit}} \sim \mathcal{F}(X)e^{-1/X^{d-1}}$$

$e^{-1/x}$ is nonanalytic at $x = 0$: all derivatives vanish, means that free energy (which has no imaginary part in stable phase) is smooth



Analyticity of F means that the imaginary

$$f(h) = \frac{1}{\pi} \int_{h'<0} \frac{\mathrm{d}h' \ \mathrm{Im}\, f(h')}{h' - h}$$