Universal scaling and the essential singularity at the abrupt Ising transition

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Outline

- Renormalization and the Ising model

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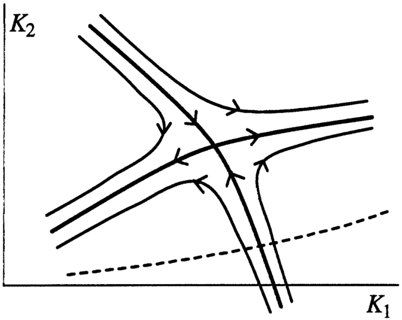
- Closed-form results for the 2D Ising susceptibility

Renormalization and the Ising Model



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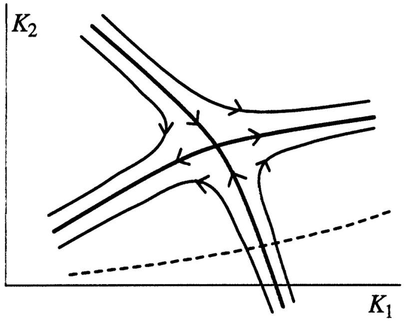

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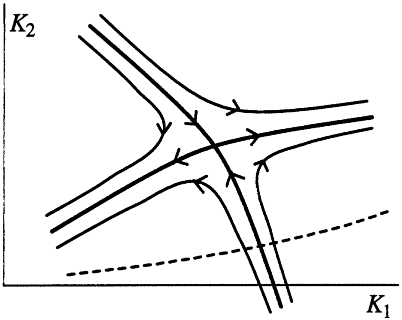

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Fixed points correspond to phases, criticality.

Nonanalytic behavior is preserved by RG.

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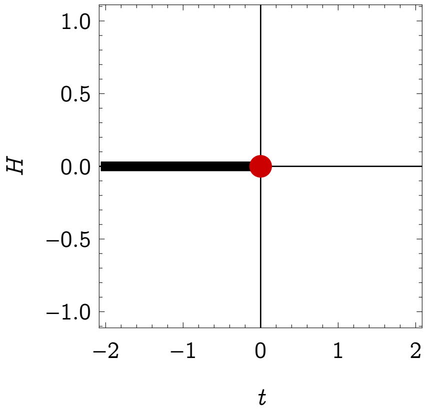

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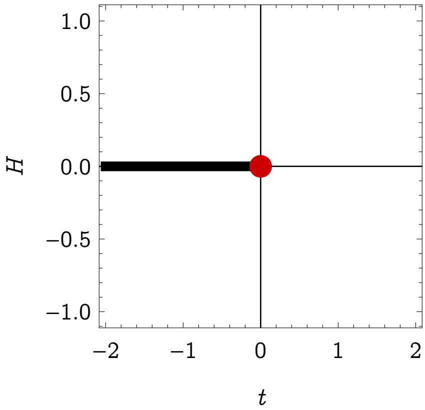

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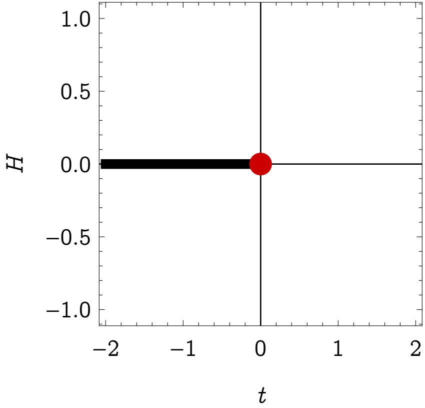

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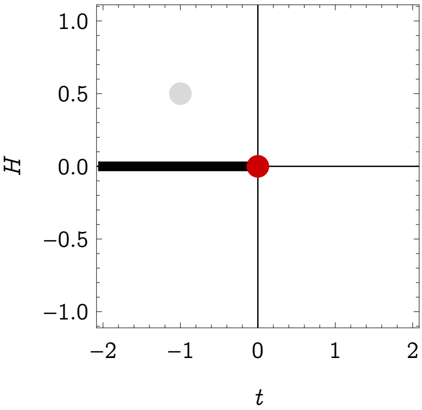
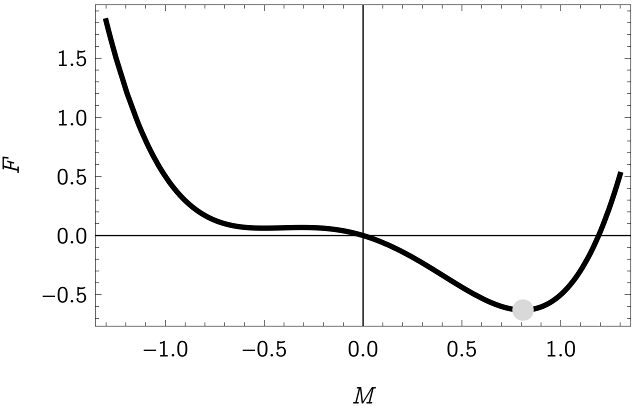
Connected to line of abrupt transitions.

We've identified nonanalytic behavior along the abrupt transition line.

Metastability & Complex Free Energy





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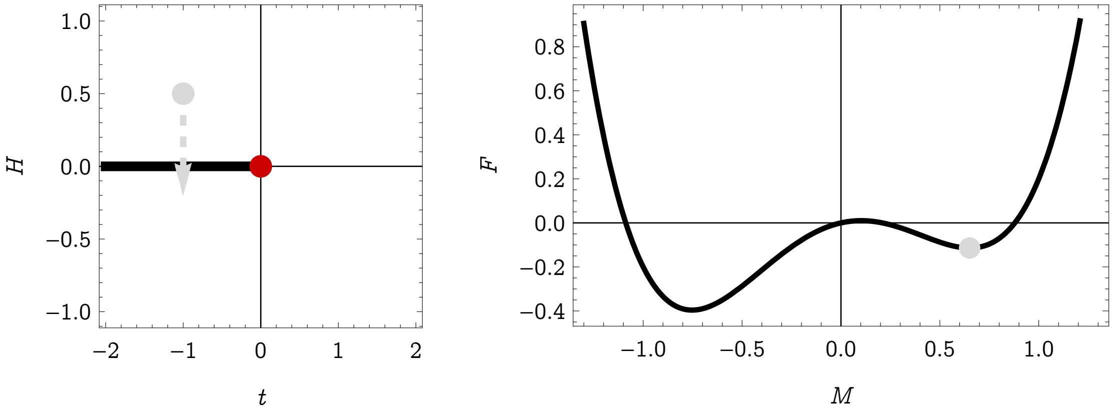

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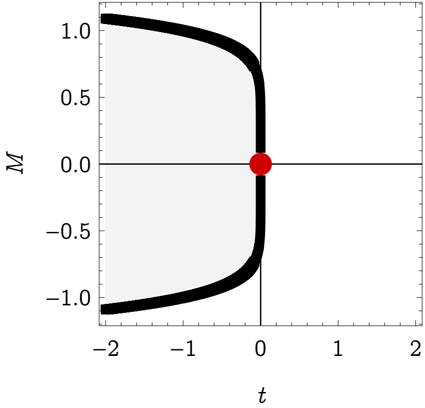
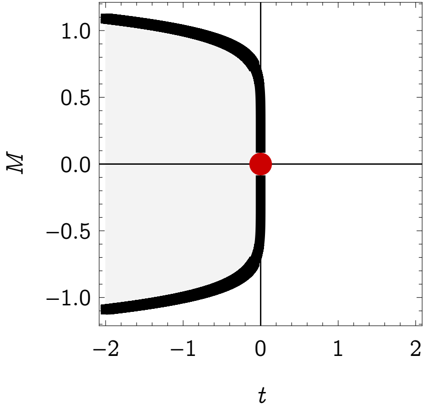


Thermodynamics can be continued into metastable phase.

Metastability & Complex Free Energy



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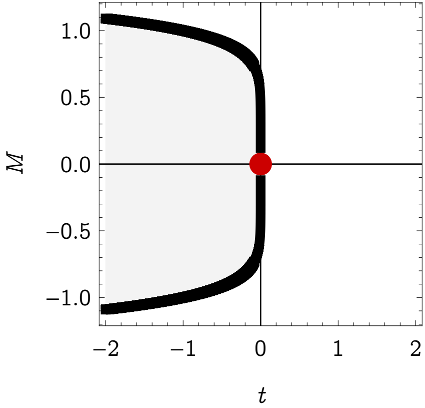

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Ising metastable decay somewhat well studied (Günther 1980, Houghton 1980)

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Metastable phase is stable to domains smaller than

$$N_{\text{crit}} = \left(\frac{MH}{\sigma\Sigma}\right)^{-1/(\sigma-1)}$$

but larger will grow to occupy the entire system, decay to stable phase.

The Metastable Ising Model

The formation of a critical domain has energy cost

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Decay of the equilibrium state implies existence of an imaginary part in the free energy,

$$\operatorname{Im} F \sim e^{-\beta \Delta F_{\text{crit}}}$$

The Metastable Ising Model

Near the Ising critical point, $\sigma = 1 - \frac{1}{d}$ and

$$M = t^{\beta}\mathcal{M}(h/t^{\beta\delta}) \qquad \Sigma = t^{\mu}\mathcal{S}(h/t^{\beta\delta})$$

with $\mathcal{M}(0)$ and $\mathcal{S}(0)$ nonzero and finite.

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Near the Ising critical point, $\sigma = 1 - \frac{1}{d}$ and

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with $\mathcal{M}(0)$ and $\mathcal{S}(0)$ nonzero and finite.

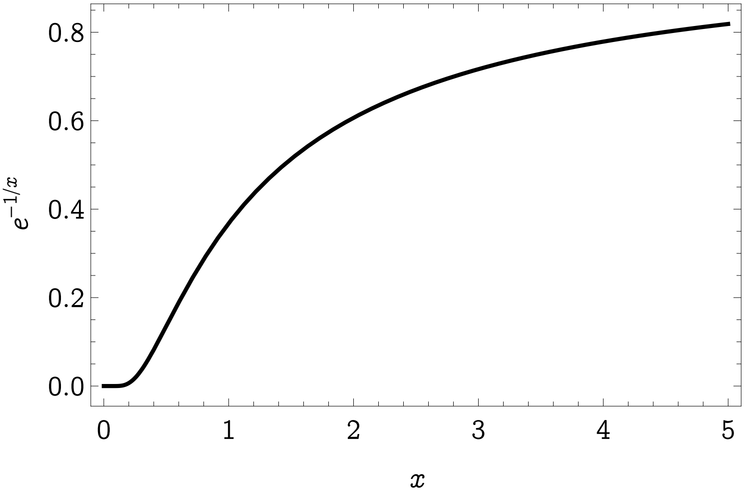
Therefore,

$$\Delta F_{\text{crit}} \sim \Sigma\left(\frac{MH}{\Sigma}\right)^{-(d-1)} = X^{-(d-1)}\mathcal{F}(X)$$

for $X = h/t^{\beta\delta}$, and

$$\operatorname{Im} F = t^{2-\alpha}\mathcal{I}(X)e^{-\beta/X^{(d-1)}}$$

The Essential Singularity



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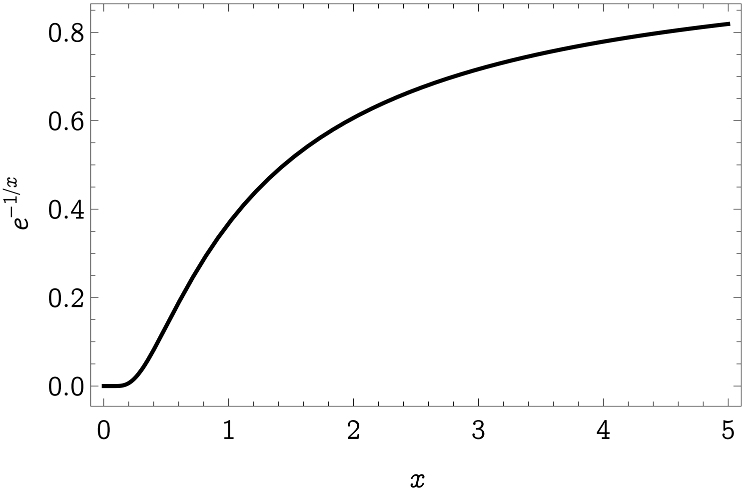

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Analytic properties of the partition function imply that

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Only predictive for high moments of F, or

$$f_n = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X'^{n+1}} \, \mathrm{d}X'$$

for $F = \sum f_n X^n$.

The Essential Singularity

Results from field theory indicate that $\mathcal{I}(X) \propto X + \mathcal{O}(X^2)$ for $d = 2$, so that

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Not a convergent series—the real part of F for $H > 0$ is also nonanalytic!

The Essential Singularity

In two dimensions, the Cauchy integral does not converge, normalize with λ,

$$F(X \mid \lambda) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(X')}{X' - X} \frac{1}{1 + (\lambda X')^2} \, \mathrm{d}X'$$

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Exact result has form

$$\begin{aligned} F(X \mid \lambda) = \frac{A}{\pi} \frac{1}{1 + (\lambda X)^2} \Big[& X e^{B/X} \operatorname{Ei}(-B/X) \\ & + \frac{1}{\lambda} \operatorname{Im}\left(e^{-i\lambda B}(i + \lambda X)(\pi + i \operatorname{Ei}(i\lambda B))\right) \Big] \end{aligned}$$

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The Cauchy integral is only predictive for high moments.

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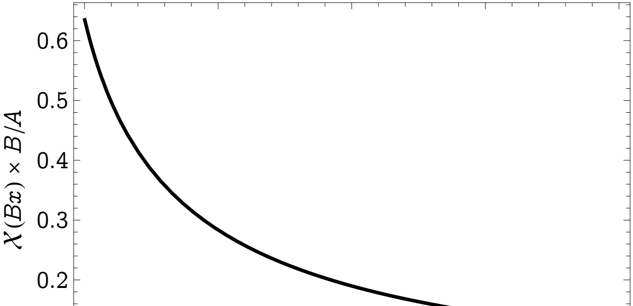
Has a well-defined limit as $\lambda \to 0$, simple functional form:

$$\chi = t^{-\gamma}\mathcal{X}(h/t^{\beta\delta})$$

where the scaling function is

$$\mathcal{X}(X) = \frac{A}{\pi X^3}\left[(B - X)X + B^2 e^{B/X}\,\mathrm{Ei}(-B/X)\right]$$



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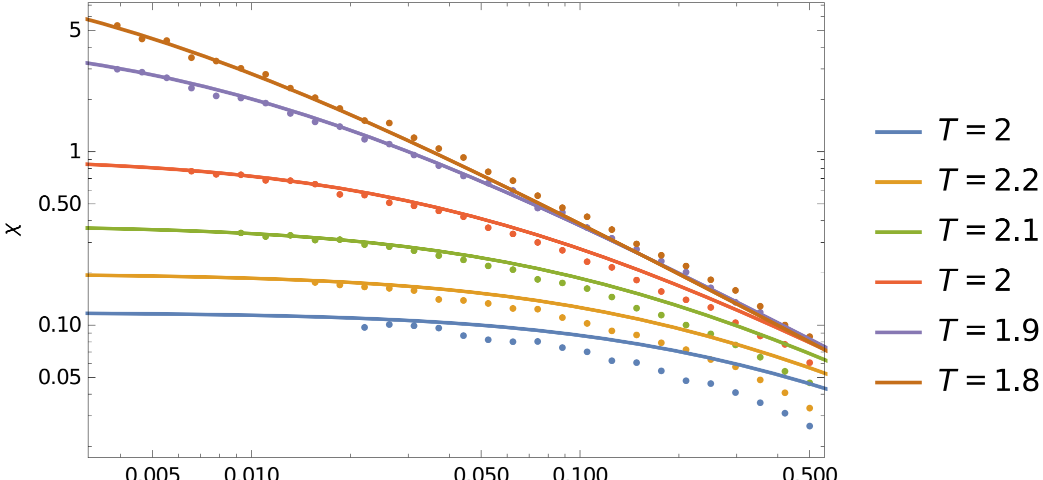

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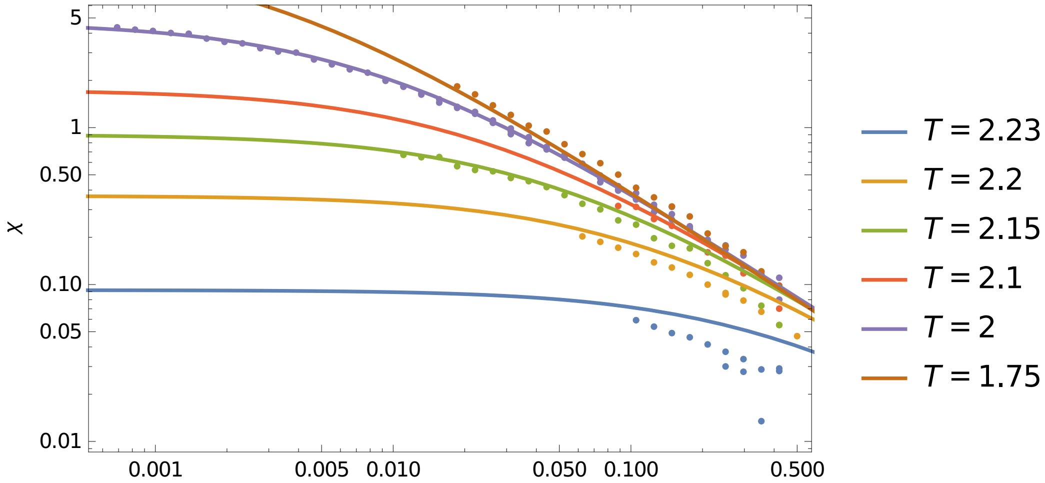

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Remain on the lookout for other universal properties to incorporate.

Questions?