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SWENDSEN–WANG SIMULATION OF ISING SPINS AND A PRECISE DEFINITION OF CRITICAL CLUSTERS

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A recent ultrafast simulation of Ising spins ($\sigma_i = \pm 1$) utilizes a random + to − flip-over of duly defined decoupled blocks of spins. We show that the *random dynamics alone* suffices to prove the correspondence of the blocks with "critical clusters" describing thermal (magnetic) fluctuation. (The precise requirement is $\sigma_i\sigma_j = 1$, for a pair of spins inside a block, and $\langle \sigma_i\sigma_{j'} \rangle = 0$, for a σ_i inside and $\sigma_{j'}$ outside.) The present approach helps to extend the study of critical clusters, and also ultrafast simulation, to the case of nonzero magnetization. Finite critical clusters constitute always a $\pm$ symmetric set and are very different (much smaller) than continuous domains of similarly oriented spins.

1. Introduction

An important problem in the study of N-body systems is that of a satisfactory definition of "critical clusters". These should describe the divergent fluctuation of the order parameter near to a critical temperature T_c. A useful delimitation of such clusters becomes of particular interest in connection to computer simulations of actual $N \gg 1$ microstates at $T \approx T_c$. Intuitively one feels that a cluster should somehow represent the effective range of pair correlations, but a precise, generally valid, formulation of that intuition is not available. A formal approach has been proposed some time ago for the Ising model of magnetic spins [1, 2]. It transforms the interaction between neighbor spins, into "bonds" which are either "frozen" or "killed" (freezing/killing transformation, FKT, in brief). This creates decoupled blocks of spins, each of which is continuously connected by frozen bonds, and is delimited by a perimeter consisting of killed bonds. The blocks correspond to the critical clusters, i.e. they represent precisely the thermal magnetic fluctuation. For awhile however, this representation of interactions by clusters enjoyed the status of a very formal theory, limited to the $n = 1$ Ising model. Renewed interest in it arose in connection to computer simulations: A conventional Monte Carlo simulation [3] at $T \simeq T_c$ is severely hampered by critical slowing

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down. Swendsen and Wang [4] have therefore proposed a superfast simulation which utilizes the concept of clusters/blocks. It applies FKT to a given Ising microstate, dividing it into decoupled clusters/blocks. The subsequent simulation flips over randomly entire blocks of spins. An improvement of the simulation, and application to other models followed [5, 6]. Still, published applications of FKT have been carried out at magnetization $\langle M \rangle = 0$ alone (viz. at $T \geqslant T_c$ and magnetic field $H = 0$). The present article extends the definition of critical clusters, and, by the same token, the superfast simulation to $\langle M \rangle \neq 0$ as follows. First, the intuitive definition of a cluster is formulated as a definite requirement on spin pairs inside and outside a cluster. Second, it is shown that *because of their random dynamics* the FKT blocks fulfill this requirement, both at $\langle M \rangle = 0$ and $\langle M \rangle \neq 0$. The consequent correspondence of FKT blocks to thermal clusters is exact. This is also demonstrated by actual computer simulation results. The present application of such a "dynamic" approach to the definition of critical clusters is expected to lead to further applications, in critical dynamics especially.

2. A definition of critical clusters

Take an Ising lattice consisting of spins i $(1, \ldots, N)$, with orientation $\sigma_i = \pm 1$. A cluster of size s may be identified with a subset of spins, s_i, defined with respect to an ith spin, as follows [7]. Spins inside the cluster, $j \not\in s_i$ (i itself included), fulfill $\sigma_i \sigma_j = 1$. Spins outside the cluster, $j' \not\in s_i$, fulfill $\langle \sigma_i \sigma_{j'} \rangle_{s_i} = 0$ (ensemble average over $\sigma_{j'}$ at constant s_i). Such a cutoff of pair correlations, if attainable, helps to calculate the average square magnetization per spin,

$$\langle M^2 \rangle / N = \left\langle N^{-1} \sum_{i=1}^{N} \left\langle \sum_{j=1}^{N} \sigma_i \sigma_j \right\rangle_{s_i} \right\rangle = \langle s_i^2 \rangle \,. \tag{1}$$

Here the outer average $\langle\ \rangle$ refers to ensemble average over s_i $(1, \ldots, N)$. The magnetic fluctuation per spin, $\Delta M^2 = (\langle M^2 \rangle - \langle M \rangle^2)/N$, becomes

$$\Delta M^2 = s_w - \langle M \rangle^2 / N \,. \tag{2}$$

($s_w = \langle s_i^2 \rangle = \Sigma_s n_s s^2$ denotes a weight average cluster size, n_s is the average number of s-clusters per spin [8].) Such a definition equates a cluster to the effective region of a fluctuation, *decoupled* (on the average) from spins outside. This contrasts a definition of clusters like in athermal percolation, viz. as domains of + spins surrounded by −, and vice versa. Clearly for such percolation-like clusters $\langle \sigma_i \sigma_{j'} \rangle_{s_i} < 0$! (The present critical clusters are synonymous with Fisher's "droplets" [9]; this term however is avoided since

they constitute fractal objects, more resembling branched polymers than droplets [7]. The percolation-like clusters mentioned here are often called "Ising clusters" [2, 10].) In view of the scaling laws [11],

$$s_w - \langle M\rangle^2/N \sim \epsilon^{-\gamma} \qquad \text{and} \qquad R \sim \epsilon^{-\nu}\,; \tag{3}$$

here R is a ("z-average" [8]) cluster radius, $\epsilon = |T - T_c|/T_c$ while γ and ν are critical exponents. The problem with eqs. (1)–(3) is that it is not trivial to delimit the clusters as required, for example to formulate an algorithm for a computer simulation which ensures that $\langle \sigma_i \sigma_{j'} \rangle = 0$ is obeyed exactly. In the author's previous study [7], critical clusters have been defined as continuous regions of $\sigma_i \sigma_j = 1$, delimited by a perimeter consisting of + and − spins at their average ratio. The definition ensures $\langle \sigma_i \sigma_{j'} \rangle = 0$, but only as a mean-field approximation. The possibility for a precise execution of $\langle \sigma_i \sigma_{j'} \rangle = 0$ is offered by FKT.

3. FKT and block dynamics

Let the Ising Hamiltonian be denoted as $E = \Sigma_{i>}\Sigma_j e_{ij}$, where $e_{ij} = -K\sigma_i\sigma_j$ is the neighbor interaction, $K = J/kT$ and J the coupling. Assumedly $H = 0$ and $T \geqslant T_c$, so that $\langle M\rangle = 0$. E is transformed into a reduced Hamiltonian, E^*, not containing e_{ij} as a variable [1, 2, 4, 5]. Thus e_{ij} is either killed (deleted from E), with probability k_{ij}, or frozen as $e_{ij} = \text{const}$, with probability $1 - k_{ij}$ (constant with respect to subsequent transitions from one microstate to another). The probability to kill e_{ij} in a given microstate α depends on its instantaneous value ($= \pm K$). Thus

$$k_{ij(\alpha)}/k_{ij(\alpha')} = \exp[e_{ij(\alpha)} - e_{ij(\alpha')}]\,. \tag{4}$$

After FKT, transitions $\alpha \rightarrow \alpha'$ obey detailed balance determined by E^*,

$$t^*_{\alpha,\alpha'}/t^*_{\alpha'\alpha} = \exp(E^*_\alpha - E^*_{\alpha'})\,. \tag{5}$$

Eqs. (4) and (5) ensure that the original detailed balance, which is determined by $\exp(E_\alpha - E_{\alpha'})$, is not perturbed by FKT. The proof is as follows [5]. Two types of transitions $\alpha \rightarrow \alpha'$ are distinguished, depending on whether e_{ij} does, or does not, vary upon the transition. First suppose that $e_{ij(\alpha)} \neq e_{ij(\alpha')}$. Such a transition is possible only if the bond has been killed. Hence the overall probability of the transition is $t_{\alpha,\alpha'} = k_{ij(\alpha)} t^*_{\alpha,\alpha'}$. Combination with eqs. (3) and (4) shows at once that the overall probability obeys the original detailed balance,

$$t_{\alpha,\alpha'}/t_{\alpha',\alpha} = \exp(E_\alpha - E_{\alpha'}) . \tag{6}$$

Second, suppose that $e_{ij(\alpha)} = \text{const}$. Now the transition may proceed by two alternative routes, with e_{ij} either killed or frozen. Since $e_{ij(\alpha)} = e_{ij(\alpha')}$, the probabilities of these two routes to occur, k_{ij} and $1 - k_{ij}$ respectively, remain the same in α and in α'. For the same reason, for both routes, $E^*_\alpha - E^*_{\alpha'} = E_\alpha - E_{\alpha'}$, whence $t^*_{\alpha,\alpha'}/t^*_{\alpha',\alpha} = \exp(E_\alpha - E_{\alpha'})$. Therefore the overall probability of the two-route transition also obeys eq. (6), which completes the proof. FKT is applied to all e_{ij} of a particular microstate. Ultimately blocks of spins are formed: Each spin in a block is connected to it by at least one frozen bond. A block is bounded by an uninterrupted perimeter of killed e_{ij}; viz. it is *decoupled* from other blocks. When block flipping dynamics is applied, e_{ij} in a perimeter and in a block follow, respectively, $e_{ij(\alpha)} \neq e_{ij(\alpha')}$ and $e_{ij(\alpha)} = \text{const}$. Both obey eq. (6) but, since the blocks are decoupled from their surrounding, $E_\alpha - E_{\alpha'} = 0$. Thus, the flip-overs of blocks, back and forth from $+s$ to $-s$, actually obey random dynamics,

$$t_{\alpha,\alpha'}/t_{\alpha',\alpha} = t_{+s,-s}/t_{-s,+s} = \text{const} , \qquad \text{for } \langle M \rangle = 0 . \tag{7a}$$

It is important to note that, in general (not for $\langle M \rangle = 0$ only), a utilization of FKT for effective block dynamics implies the fulfillment of two more conditions. First, it is convenient (not necessary) to have pure blocks, consisting of $+(-)$ spins alone. Hence (replacing subscript ij by $+-$, for the case $\sigma_i \neq \sigma_j$), we require $k_{+-} = 1$. Second, blocks should be preserved during their flip-overs from $+$ to $-$ and vice versa. Hence the e_{ij} terms in the Hamiltonian that undergo FKT have to fufill

$$e_{++} = e_{--} \Rightarrow k_{++} = k_{--} \quad [= \exp(-2K)] . \tag{8}$$

Indeed the proof of eq. (6) for the relevant case, $e_{ij(\alpha)} = \text{const}$, requires that. Thus, even if $H \neq 0$ and the Hamiltonian is $E^H = E - H\Sigma_i \sigma_i$, FKT still has to obey eq. (8). Consequently the magnetic field term for each spin involved in the block flip-overs, $H\sigma_i$, has to be absorbed into $t_{\alpha,\alpha'}/t_{\alpha',\alpha}$ and *not* into k_{++}/k_{--}.

4. Spin blocks describe critical clusters

Take $\langle M \rangle = 0$ first. The mutually independent dynamics of spins inside and outside a block (eq. (7a)), establishes $\langle \sigma_i \sigma_{j'} \rangle = 0$, while $\sigma_i \sigma_j \equiv 1$. This agrees with the requirements set in eq. (1), so that the spin blocks precisely represent

the "critical" clusters. It should be noted that the division of a given lattice microstate is stochastic, i.e. not unique. The random dynamics makes the set of blocks $+-$ symmetric, $n_{+s} = n_{-s}$. This of course agrees with $\langle M \rangle = 0$, and for that reason $\langle M \rangle^2$ makes no contribution in eqs. (2), (3). Let this be now extended to $\langle M \rangle \neq 0$.

To begin with take the $+-$ symmetric cold case, $T < T_c$ and $H = 0$. Since $E_{+s} = E_{-s}$ remains valid, the flip-overs should still be random (eq. (7a)). However, random flip-over of blocks unhindered by critical slowing converges upon $\langle M \rangle = 0$, wiping out any initial state $\langle M \rangle \neq 0$. (In this respect the conventional Monte Carlo simulation of individual interacting spins, which is slowed down and does not converge to $\langle M \rangle = 0$ for sufficiently large N, provides a better simulation of macroscopic behavior!) To solve the problem we take a hint from the cluster theory of critical phenomena [8]. For the case $\langle M \rangle \neq 0$ the theory divides the set of clusters, into an "infinite" cluster and a remaining subset of "finite" clusters. The former alone gives rise to symmetry breaking, viz. $t_{+\text{infi},-\text{infi}} = t_{-\text{infi},+\text{infi}} = 0$, conserving $\langle M \rangle = \langle M_{\text{infi}} \rangle \neq 0$. The subset of finite clusters is symmetric, $\langle M_{\text{fini}} \rangle = 0$. However $\langle M^2_{\text{fini}} \rangle$, together with $\langle M^2_{\text{infi}} \rangle - \langle M_{\text{infi}} \rangle^2$, gives rise to magnetic fluctuation ΔM^2 (ref. [8] approximates the fluctuation by $\langle M^2_{\text{fini}} \rangle$ alone). In accordance, the following algorithm is adopted here. FKT is applied just as before (eqs. (4, 5)). The resultant set of decoupled blocks is subjected to the random dynamics of eq. (7a). However, in view of the hint taken from critical theory, the largest ("infinite")+ block on a given latice, is singled out to be kept constant. Formally

$$t_{+s,-s} = \begin{cases} t_{-s,+s} = \text{const}, & \text{if "finite"}, \\ 0, & \text{if "infinite"}, \end{cases} \qquad \text{for } T < T_c \text{ and } H = 0. \quad (7b)$$

This restricted block dynamics is followed by a reapplication of FKT; a new largest + spin block is singled out, etc., until equilibrium. For large enough N and at $T < T_c$, we should "almost always" obtain a microstate dominated by a $s_+ = \mathcal{O}(N)$ block, i.e. the process converges upon the $M > 0$ half of the phase space. Repeated renewal of FKT should ensure uniform visiting of this half-space. The problem of correct block dynamics is thereby solved. (Its application speeds up convergence while the finite blocks are still relatively large, otherwise the gain is insignificant, but in that case a Monte Carlo simulation of single spins becomes anyhow adequate.) As to the correspondence of blocks to critical clusters: The blocks are decoupled as before. The independent motion of the finite blocks establishes $\langle \sigma_i \sigma_{j'} \rangle = 0$, both when i and j' belong to two different finite blocks, or when one belongs to a finite and the other to the immobile infinite block. Eq. (1) is therefore fulfilled. Furthermore, since the subset of finite clusters is $+-$ symmetric (eq. (7b)),

$\Sigma_{s,\text{fini}}(n_{+s} - n_{-s})s = 0$, whence $\langle s_{\text{infi}} \rangle = \langle M \rangle / N$. Concerning the fluctuation, it is important to realize that both the subset of the finite clusters and s_{infi} contribute to $\langle M^2 \rangle$ in eq. (1). Eq. (2) becomes

$$\Delta M^2 = s_{\text{w,fini}} + s_{\text{w,infi}} - \langle s_{\text{infi}} \rangle^2 . \tag{9}$$

The correlation length (eq. (3)) is identified with R of the finite clusters.

Let us turn to the asymmetric case $H \neq 0$. As has been pointed out in eq. (8), the correct extension of FKT requires that we still have $k_{++} = k_{--}$ $(= \exp - 2K)$, the H dependence being absorbed into the ratio of transition probabilities. Thus the equation of motion (eq. (7a)) becomes

$$t_{+s,-s}/t_{-s,+s} = \exp(-2sH) , \qquad \text{for } H > 0 , \tag{7c}$$

which once again solves the problem of correct dynamics. The correspondence of blocks to clusters is more intricate. At equilibrium $n_{-s} = n_{+s}\exp(-2sH)$. Hence n_{-s} and the fraction $n_{+s}\exp(-2sH)$ constitute together a $\pm$ flipping subset of finite clusters of average magnetization, $\Sigma_s[n_{+s}\exp(-2sH) - n_{-s}]s = 0$. The remaining + blocks describe the "infinite" cluster (somewhat abstractly since they are not connected!), which is immobile and gives $\langle s_{\text{infi}} \rangle = \Sigma_s n_{+s}[1 - \exp(-2sH)]s = \langle M \rangle / N$. Just as before, the independent motion of the finite blocks establishes $\langle \sigma_i \sigma_{j'} \rangle = 0$, both when i and j' belong to two finite clusters, or when one belongs to a finite and the other to the immobile infinite cluster. Eq. (1) is therefore fulfilled. It would not be fulfilled if i and j' could belong to two *separate* infinite clusters. For that reason we define a single, not necessarily connected, s_{infi} and s^2_{infi}. (The calculation of the latter is affected by the required singleness; incidentally, a cluster's connectedness constitutes a gratifying but not a necessary condition.) In detail,

$$s_{\text{w,fini}} = \sum_s [n_{-s} + n_{+s}\exp(-2sH)]s^2$$

and (10)

$$s_{\text{w,infi}} = \langle \{ \sum_s n'_{+s}[1 - \exp(-2sH)]s \}^2 \rangle .$$

Here n'_{+s} denote instantaneous numbers for a microstate. The magnetic fluctuation is calculated from eqs. (9) and (10); R, as before, refers to finite clusters. The absorbtion of the H dependence into the transition probability (eq. (7c)) has been already proposed in ref. [4] but, since blocks are not actually compared with the magnetic fluctuation, the precise division into $s_{\text{w,fini}}$ and $s_{\text{w,infi}}$ (eq. (10)) is not elaborated. It should be pointed out that although the modified evaluations of the magnetic fluctuation introduced in eqs. (9) and

(10) may seem to constitute a trivial detail, they contribute a factor of two and more to the numerical comparison of blocks with the magnetic fluctuation, in the next section. Ref. [1] proposes to carry out FKT with a dummy spin (connected to all others), representing the H dependence. Although the blocks dynamics is not studied, the proposal implies a division of blocks into randomly moving $s_{w,fini}$, and one fixed (disjoined) $s_{w,fini}$ connected to the dummy spin; this would lead to the same result as here.

5. Simulation results for square Ising lattice

The cold $T < T_c$ ($H = 0$) and the $H > 0$ ($T = T_c$) cases are displayed in fig. 1. A point represents a run of 200–1000 lattice cycles, each consisting of a flip-over of all movable blocks, followed by 5 Monte Carlo sweeps of all spins

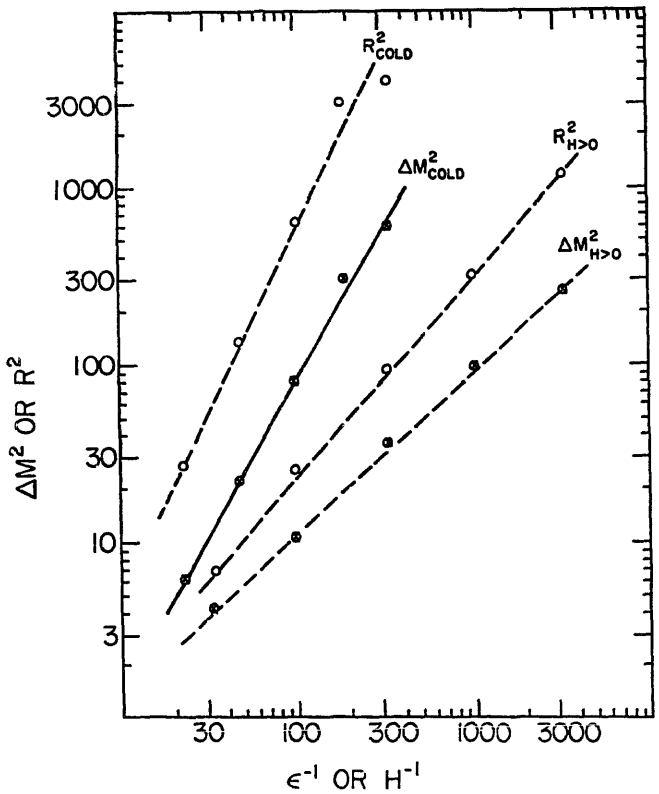

Fig. 1. Magnetic fluctuation ΔM^2 and cluster square radius R^2 are plotted vs. ϵ^{-1} or vs. H^{-1}, for the cold and $H > 0$ cases, respectively. H is in units of K. Directly measured ΔM^2, and ΔM^2 calculated from the weight average cluster size s_w with the help of eqs. (2), (9) and (10), are denoted by (×) and by (○), respectively. The two sets of points almost coincide. Solid line describes exact results for cold ΔM^2. Broken lines drawn through the experimental points for ΔM^2 at $H > 0$, and for R^2 (also denoted by ○), give $R^2 \sim \epsilon^{-2.03}$, $\Delta M^2 \sim H^{-0.90}$ and $R^2 \sim H^{-1.12}$, which agree to within experimental error with the theoretical exponents (see text).

(the lattice size being 200^2–400^2). The results demonstrate that the weight average cluster size, s_w, minus $\langle M^2 \rangle / N$ and the directly measured magnetic fluctuation ΔM^2 are indistinguishable to within experimental accuracy (eq. (2)). The slopes of the logarithmic plots also agree well with the theoretic $\Delta M^2 \sim \epsilon^{-\gamma} = \epsilon^{-7/4}$ and $R^2 \sim \epsilon^{-2\nu} = \epsilon^{-2}$, and with $\Delta M^2 \sim \langle M \rangle^{-\gamma/\beta} \sim H^{-\gamma/\beta\delta} = H^{-14/15}$ and $R^2 \sim H^{-2\nu/\beta\delta} = H^{-16/15}$, for the cold and $H > 0$ cases, respectively. The agreement supports the present extension of superfast simulations.

6. Conclusions

"Critical clusters" should describe the divergent fluctuation of the order parameter near a critical temperature T_c, viz. their square size minus $\langle M^2 \rangle / N$ and radius should scale with exponents γ and ν respectively (eq. (3). Taking Ising spins as an example, this leads to the requirement $\sigma_i \sigma_j = 1$, for a pair of spins inside a block, and $\langle \sigma_i \sigma_{j'} \rangle = 0$, for a σ_i inside and $\sigma_{j'}$ outside (eq. (1)). The definition of a cluster by this requirement has been proposed before [7], but its fulfillment, with the help of a "perimeter of average composition", was approximate. At present it is fulfilled precisely by the decoupled blocks of similar spins created with the help of the Fortuin–Kasteleyn freezing/killing transformation (FKT). The correspondence of the FKT blocks, to the critical clusters, has been known for some time [1, 2]. Here it is proven with the help dynamic reasoning: A ± flip-over of the decoupled blocks is random and this immediately leads to $\langle \sigma_i \sigma_{j'} \rangle = 0$, as stipulated. The correspondence holds for the one-phase case, $\langle M \rangle = 0$ at $T \geq T_c$ and $H = 0$; for the two-phase case $\langle M \rangle \neq 0$ at $T < T_c$ (with the largest + block kept fixed); and for the asymmetric $H \neq 0$ case (with H absorbed into flip-over probability). It transpires that the accepted concept of clusters [8] needs to be modified: (a) when $\langle M \rangle \neq 0$, the infinite cluster also contributes to ΔM^2 (eq. (9)); (b) when $H > 0$, the infinite cluster is disjoined, consisting of the immovable fraction of the + clusters (eq. (10)); and (c) under all conditions, the random flip-over of the finite clusters implies their complete +− symmetry, in number and shape.

Swendsen and Wang [4] have utilized the random flip-over of the FKT blocks in order to devise a superfast simulation. Our generally valid cluster to block correspondence enables us to extend the superfast simulation to the two phase case $T < T_c$ (eq. (7b)). The relative simplicity of the present dynamic reasoning might help to devise a superfast simulation, and to identify critical clusters, in other problems involving many-body interaction.

The determination of a set of criticial clusters is not unique, but stochastic (FKT). They do not therefore represent any concrete object, but rather a distribution of interacting particles. One can ask therefore whether their

precise definition and simulation leads to new information. It seems that the study of fractal and "spectral" dimensions of clusters, and mass–length dependence (exponents d_f, d_s and γ_t, respectively, in ref. [7]), of their nucleation and fragmentation [12, 13], and of the relation between their structure to critical slowing down [14], testify to their physical significance. (Incidentally, the values of exponents d_f and γ_t, determined for the approximately defined clusters in ref. [7], are quite similar to those determined for the present exactly defined clusters, cf. ref. [14]). It is important to realize that the critical clusters are very different from the geometric clusters, corresponding to continuous domains of + (−) spins, which describe percolation. The latter are mutually repelling (for the ferromagnetic interaction), whereas the decoupled critical cluster/blocks constitute (on the average) ideal supermolecules! The profound shape difference between the two is illustrated in fig. 2; (a) vs. (b), respectively, for $T > T_c$, and (c) vs. (d), respectively, for $T < T_c$. Fig. 2a shows how intermixed the critical clusters are; the corresponding continuous regions of + (−) in (b) enclose one another, which clearly disagrees with the requirement that clusters at $T > T_c$ constitute *finite fractals*. Fig. 2c displays the existence of

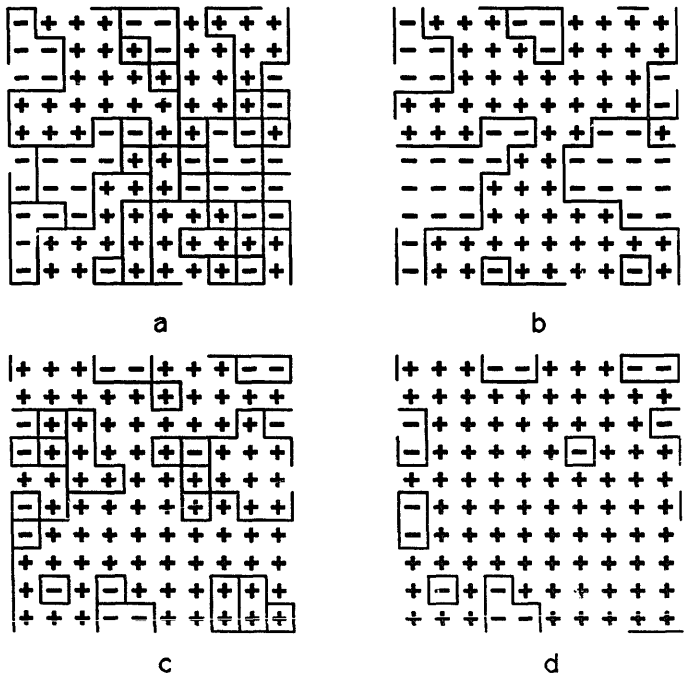

Fig. 2. A comparison of critical clusters, defined by FKT, with percolation-like ones, defined as continuous regions of + (−) spins. Solid lines denote boundaries of clusters (which may carry over an edge due to periodic conditions). Figures on the left describe a distribution of critical clusters obtained on a typical 10×10 Ising lattice, (a) and (c), respectively, at T slightly above and below T_c. Figures on the right (obtained from those on the left by erasing the boundaries between critical clusters of the same sign), describe the corresponding continuous regions of + and − spins, with (b) and (d), derived from (a) and (c), respectively.

large + clusters imbedded in an infinite + cluster; this should show up for example in a fragmentation of a corresponding many-body system along "easy lines" of effective decoupling. The following perhaps bizarre reflection is also worth noticing. The Ising model describes the gas–liquid transition [15]. The ± symmetric finite clusters consist then either of molecules, or of empty spaces; viz. vacuum too is organized into fractals.

Note added in proof

Following the submission of this article to Physica A, preprints of related articles by J. Kertész [16], by J.S. Wang [17], and by A. Coniglio et al. [18], have reached the author. A clarification of the present status of "critical clusters" seems appropriate. The identification of critical clusters with FKT blocks of spins appears to encounter two problems at $\langle M \rangle \neq 0$. First, for magnetic field $H > 0$, one can locate a percolation transition line $T^*(H)$, which separates a low-temperature percolative and a high-temperature non-percolative region [19]. The line appears to disagree with thermodynamic behavior of the Ising model, which exhibits no transition at $H \neq 0$ [20]. This difficulty is removed by the present definition of the "infinite" cluster s_{infi} ($\rightarrow \langle M \rangle \neq 0$) for $H > 0$. Thus in eq. (10), s_{infi} is identified with a set of the immobile (possibly disconnected) + clusters. For $H > 0$ and an infinite system, the number of spins in this set is always infinite. The transition at $T^*(H)$ merely indicates that one of the clusters that together make up an infinitely large s_{infi}, becomes by itself infinitely large, clearly a geometric and not a thermodynamic property. (Indeed $T^*(H)$ seems to describe another geometric transition, in the distribution and shape of finite clusters [16].) The same explanation obtains with the help of a ghost-spin definition of s_{infi} [1], as is pointed out in refs. [17] and [18]. The other problem relates to the critical behavior in the cold region at $H = 0$, as T approaches T_c from below. A mean field calculation [21] has shown that the weight average size of the finite FKT clusters, $s_{\text{w,fini}}$, diverges with an exponent different from exponent γ, which describes magnetic fluctuation ΔM^2. The difficulty is also resolved at once by the present approach. Thus, in eqs. (1)–(3), ΔM^2 is equated to s_w of *all* clusters, finite and infinite alike, minus $\langle M \rangle^2/N$ $(= \langle s_{\text{infi}} \rangle^2)$. Present simulation results support this equality, with $s_{\text{w,fini}}$ and $\Delta M^2_{\text{infi}} = s_{\text{w,infi}} - \langle s_{\text{infi}} \rangle^2$ providing comparable contributions to ΔM^2. The ΔM^2_{infi} contribution to ΔM^2 has been found before, but for $H = 0$ only [22, 20] (a factor 1/4 appears due to an identification of finite clusters with blocks of down spins alone). With the present approach the contribution of ΔM^2_{infi} is evident at all H and T (it is zero for $\langle M \rangle = 0$.) The same conclusion is

reached in refs. [17, 18]. An intriguing open question concerns the accepted identification of the correlation length with R measured over the set $s_{w,fini}$. If the scaling of ΔM^2_{infi} happens to be different and to dominate that of $s_{w,fini}$ (a case not found in our simulations), then R should be related to ΔM^2_{infi} instead.

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