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Dynamic critical exponents for Swendsen–Wang and Wolff algorithms obtained by a nonequilibrium relaxation method

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Abstract. Using a nonequilibrium relaxation method, we calculate the dynamic critical exponent z of the two-dimensional Ising model for the Swendsen–Wang and Wolff algorithms. We examine dynamic relaxation processes following a quench from a disordered or an ordered initial state to the critical temperature T_c , and measure the exponential relaxation time of the system energy. For the Swendsen–Wang algorithm with an ordered or a disordered initial state, and for the Wolff algorithm with an ordered initial state, the exponential relaxation time fits well to a logarithmic size dependence up to a lattice size $L = 8192$. For the Wolff algorithm with a disordered initial state, we obtain an effective dynamic exponent $z_{\text{exp}} = 1.19(2)$ up to $L = 2048$. For comparison, we also compute the effective dynamic exponents through the integrated correlation times. In addition, an exact result of the Swendsen–Wang dynamic spectrum of a one-dimensional Ising chain is derived.

Keywords: classical Monte Carlo simulations, critical exponents and amplitudes (theory), finite-size scaling

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1. Introduction

In the last two decades, cluster algorithms have played an important role in statistical physics due to their reduced critical slowing down, improved computational efficiency, and interesting dynamical properties. Among these dynamical properties, the dynamic critical exponent z , which describes the divergent correlation time, is the centre of attraction.

There are various ways to calculate the dynamic critical exponent z —for example, through the exponential decay of the time correlation of a finite system in equilibrium [1, 2], or from the dynamic scaling behaviour in nonequilibrium states [3]–[5]. In calculating the time correlation in equilibrium, the difficulty is that one can hardly reach a very large lattice. The advantage for computing the dynamic exponent from a nonequilibrium relaxation process is that the finite size effect is more or less negligible, since the spatial correlation length is small in the early stages of the dynamic relaxation. Such a nonequilibrium approach, however, becomes subtle for the cluster algorithms, for the dynamic exponent z is believed to be close to zero. In addition, it is also somewhat controversial in defining a Monte Carlo time for the Wolff algorithm.

In a recent article [6], an attempt is made to estimate the dynamic exponent z of the Wolff algorithm from the finite size scaling behaviour in a nonequilibrium state. A vanishing z value is claimed. In our understanding, however, the identification of the dynamic scaling behaviour there seems to be not appropriate. On the other hand, although the cluster algorithms are known to be very efficient in reducing critical slowing down with a small dynamic exponent z , it has not been rigorously studied what precise values the dynamic exponent z takes for different variants of the algorithms. This is important in theory and application of the cluster algorithms.

In this paper, we will calculate the dynamic exponent z for the Wolff [7] and Swendsen–Wang [8] algorithms, respectively through the exponential relaxation time and integrated correlation time [9] of the system energy in nonequilibrium relaxation processes. Compared with methods based on computations of time correlation functions in equilibrium, much larger system sizes can be reached in the nonequilibrium dynamic approach, especially in the case of the two-dimensional Ising model, where the system

Dynamic critical exponents for Swendsen–Wang and Wolff algorithms obtained by a nonequilibrium relaxation method energy in the equilibrium state is known exactly. Compared with the methods in [6], the system energy in our calculations is self-averaged and thus much less fluctuating in simulations of large lattices.

The paper is organized as follows. In section 2, the general theory of the spectrum of the Monte Carlo dynamics is described, and in section 3, an exact calculation of the spectrum of the one-dimensional Ising chain is formulated. In section 4, the simulation set-up is discussed in detail, and in section 5, numerical results are presented.

2. Spectrum of Monte Carlo dynamics

In order to justify our method, we first look at the spectrum of the Markov chain Monte Carlo dynamics [10] and its relation to observables, i.e., the equilibrium and nonequilibrium relaxation functions. Let W be a transition matrix of an irreducible, aperiodic, and reversible Markov chain with an equilibrium (invariant) probability distribution p . We have a detailed balance equation between W and p ,

$$p_i W_{ij} = p_j W_{ji}. \quad (1)$$

This equation implies that the following matrix is symmetric:

$$S_{ij} = p_i^{1/2} W_{ij} p_j^{-1/2}, \quad (2)$$

and, thus, the eigenvalues λ_m of S are real. Due to the conservation of the total probability, it can also be shown that $|\lambda_m| \leq 1$. Let the eigenvectors of S be u_{im} for eigenvalue λ_m ; then the left and right eigenvectors of W are $x_i = p_i^{1/2} u_{im}$ and $y_i = p_i^{-1/2} u_{im}$, respectively, such that

$$xW = \lambda_m x, \quad Wy = \lambda_m y. \quad (3)$$

The equilibrium distribution corresponds to $\lambda_0 = 1$, $x^{(0)} = p$, and $y_i^{(0)} = 1$. The next eigenvalue λ_1 nearest to 1 controls the rate of convergence. We define the exponential relaxation time τ (in units of one Monte Carlo step or attempt) by $\lambda_1 = \exp(-1/\tau)$.

We can represent the relaxation of a general observable Q in terms of the initial distribution $p(0)$ or equilibrium distribution p and the eigenspectrum of S as

$$\langle Q(t) \rangle_{p(0)} = \sum_k \lambda_k^t d_k c_k, \quad (4)$$

$$\langle Q(t)Q(0) \rangle_{\text{eq}} = \sum_k \lambda_k^t c_k^2, \quad (5)$$

where the averages are over the initial distribution and equilibrium distribution, respectively, and

$$c_k = \sum_i p_i^{1/2} Q_i u_{ik}, \quad (6)$$

$$d_k = \sum_i p_i^{-1/2} p_i(0) u_{ik}. \quad (7)$$

We define the normalized relaxation function $f(t)$ to be linear in $\langle Q(t) \rangle$ or $\langle Q(t)Q(0) \rangle$ such that $f(0) = 1$ and $f(\infty) = 0$, e.g., $f(t) = (\langle Q(t) \rangle - \langle Q(\infty) \rangle) / (\langle Q(0) \rangle - \langle Q(\infty) \rangle)$.

The integrated correlation time is defined as

$$\tau_{\text{int}} = \sum_{t=0}^{\infty} f(t). \quad (8)$$

We note that the integrated correlation time depends not only on the observable Q but also on the dynamics and the full eigenspectrum. The integrated correlation time for the equilibrium correlation and nonequilibrium relaxation is not the same. On the other hand, the exponential relaxation time, defined in the large time limit, $f(t) \sim \exp(-t/\tau)$, is an intrinsic property of the Markov chain. It is the same for both the equilibrium and nonequilibrium situations.

3. Exact calculation in one dimension

It is instructive to look at the eigenspectra of the Swendsen–Wang and Wolff dynamics in one dimension (1D). We consider the Swendsen–Wang dynamics in 1D with an open boundary condition. Consider spin σ_i at a one-dimensional lattice site $i = 1, 2, \dots, L, L+1$. We define the link variable $b_i = 1 - \delta_{\sigma_i, \sigma_{i+1}}$. The energy of the system is

$$E(\sigma) = -J \sum_{i=1}^L \sigma_i \sigma_{i+1} = 2J \sum_{i=1}^L b_i + \text{const.} \quad (9)$$

We introduce the bond variables $n_i = 0, 1$ representing the absence or presence of a bond in the Swendsen–Wang dynamics; then the joint probability distribution of the spin and bond is proportional to

$$P(\sigma, n) = \prod_{i=1}^L [p \delta_{\sigma_i, \sigma_{i+1}} \delta_{n_i, 1} + (1-p) \delta_{n_i, 0}], \quad (10)$$

where $p = 1 - \exp[-2J/(k_B T)]$. The marginal distribution of the spins is given by $\sum_n P(\sigma, n) = P(\sigma) = \prod_i \exp[-2Jb_i/(k_B T)]$. The distribution of the bonds is a special case of the Fortuin–Kasteleyn formula, $P(n) = \sum_{\sigma} P(\sigma, n) = p^{N_b} (1-p)^{L-N_b} 2^{L-N_b+1}$, $N_b = \sum_i n_i$. The Swendsen–Wang algorithm involves applying alternately two conditional probabilities, for sample bonds given the spins $P(n|\sigma) = P(\sigma, n)/P(\sigma)$, and sample spins given the bonds $P(\sigma|n) = P(\sigma, n)/P(n)$. Instead of using the spin variables, it is more convenient to use the link variable b_i . In terms of b_i , the conditional probabilities are simple products:

$$P(n|b) = \prod_i f(b_i, n_i), \quad f = \begin{pmatrix} 1-p & p \\ 1 & 0 \end{pmatrix}, \quad (11)$$

$$P(b|n) = \prod_i g(n_i, b_i), \quad g = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}. \quad (12)$$

The probability of transition from a given link configuration to another link configuration is

$$W(b \rightarrow b') = \sum_n P(b'|n) P(n|b) = \prod_i w_{b_i, b'_i}, \quad w = \begin{pmatrix} (1+p)/2 & (1-p)/2 \\ 1/2 & 1/2 \end{pmatrix}. \quad (13)$$

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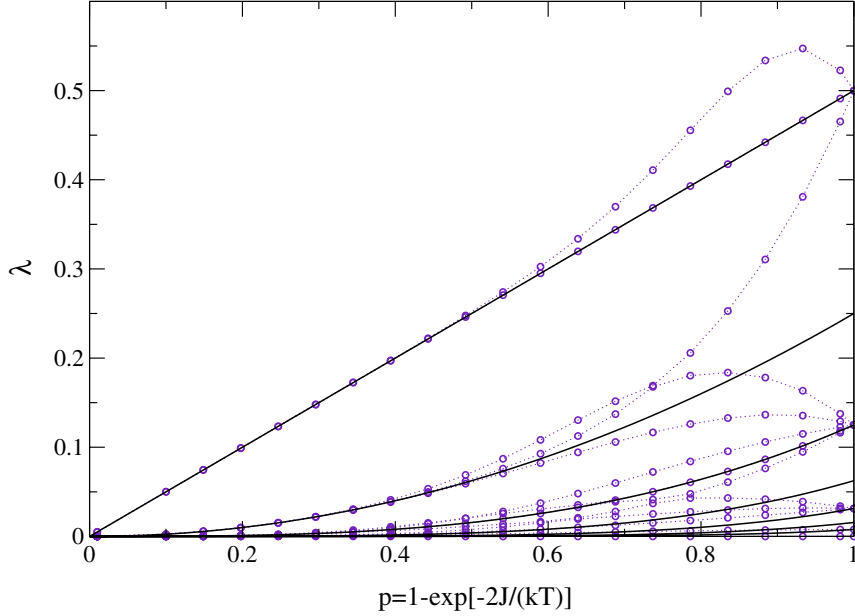


Figure 1. Eigenvalues of the transition matrix of the Swendsen–Wang dynamics in the 1D Ising chain with the open boundary condition (solid lines) and the periodic boundary condition (dotted line with circles) for a lattice size $L = 8$.

The matrix w has eigenvalues 1 and $p/2$, with left eigenvectors $v^{(1)} = (1, 1 - p)$ and $v^{(2)} = (1, -1)$, respectively. We note that the full matrix W for the whole system is a direct product of the contributions of each of the sites. Thus, the eigenvalues of W are $\lambda_m = (p/2)^m$ [11], with $L!/(m!(L - m)!)$ -fold degenerate eigenvectors, $\prod_i v^{(k_i)}(b_i)$, $k_i = 1$ or 2, with $L - m$ terms of choices for $k_i = 1$ and m terms for $k_i = 2$, where $m = 0, 1, 2, \dots, L$. The eigenvalue $\lambda_0 = 1$ corresponds to the equilibrium state with a left eigenvector $P(\sigma)$. The next eigenvalue $\lambda_1 = p/2 = \exp(-1/\tau)$ gives the relaxation time. We note that the rest of the decay times τ/m are well spaced.

It is easy to write down the probability distribution of the domain length since each link evolves independently. Let $P_l(t)$ be the probability for observing a domain of a length l with $-$ spins terminated by $+$ spins; then

$$P_l(t) = p(b = 1, t)[1 - p(b = 1, t)]^{l-1}, \quad (14)$$

where $p(t) = p(0)w^t$ is the probability that the link variable takes the value 1 at step t . A similar result, using a continuous time dynamics, is given in [12].

When a periodic boundary condition is used, we are no longer able to find the eigenspectrum analytically. In figure 1, we show the numerical results by diagonalizing the transition matrix on an $L = 8$ chain with the periodic boundary condition and compare with the open boundary condition result. We can make several interesting observations. The eigenvalue $\lambda_1 = p/2$ is always present for both periodic and open boundary conditions for any lattice size L . However, we are unable to prove this rigorously. With the periodic boundary condition, the eigenvalue $p/2$ no longer corresponds to the slowest mode. It seems reasonable that the simple spectrum $(p/2)^m$ is a good approximation if the correlation length $\xi \sim \exp(2J/(k_B T))$ is much smaller than the system size L ; thus it

Dynamic critical exponents for Swendsen–Wang and Wolff algorithms obtained by a nonequilibrium relaxation method is the correct spectrum in the thermodynamic limit. For finite sizes when ξ is comparable to size L , the degenerate spectrum splits and rejoins at $T = 0$.

4. Simulation set-up

The two-dimensional (2D) Ising model is prepared initially in a random state with a zero magnetization or at the ground state, but evolves at the critical temperature. After a certain time before reaching equilibrium, one can observe an exponential decay of the system energy, which satisfies the following equation:

$$E(t) \approx Ae^{-t/\tau} + E(\infty), \quad (15)$$

where τ is the so-called exponential relaxation time. The exponential relaxation time τ is an intrinsic property of the Monte Carlo algorithm, which is defined by the first excited eigenvalue λ_1 of the transition matrix, and should be independent of the initial states in the simulations. At the transition temperature, the dynamic scaling theory predicts that τ diverges according to $\tau \sim L^{z_{\text{exp}}}$ in the thermodynamic limit. This defines the exponential dynamic critical exponent z_{exp} . On the other hand, the integrated correlation time is defined as [9]

$$\tau_{\text{int}} = \sum_{t=0}^{\infty} \frac{E(t) - E(\infty)}{E(0) - E(\infty)}. \quad (16)$$

Similarly, one may define the integrated dynamic critical exponent z_{int} through $\tau_{\text{int}} \sim L^{z_{\text{int}}}$. In order to obtain a more accurate result, we use the exact value of $E(\infty)$ for the 2D Ising model [13].

In general, τ_{exp} and hence z_{exp} are intrinsic properties of the transition matrix. In particular, z_{exp} should be the same in equilibrium and nonequilibrium states, although the latter must be delicate: a time window must be found where the relaxation is dominated by the slowest mode. However, z_{int} depends not only on the transition matrix, but also on the observable and the initial states; hence it is rather nonuniversal.

The Wolff algorithm exhibits an important difference compared to other algorithms. It only updates the spins belonging to a certain cluster around the seed spin at each Monte Carlo step, while other algorithms sweep the whole lattice. For a fair comparison with the Swendsen–Wang or single-spin-flip algorithms, we need to rescale the Wolff Monte Carlo steps. Specifically, the Monte Carlo time t' in the Wolff algorithm should be transformed to t :

$$t = \sum_{t''=1}^{t'} \frac{C(t'')}{L^d} \quad (17)$$

where t' or t'' is the Monte Carlo time step of the Wolff single-cluster flip (i.e. the number of clusters flipped so far), $C(t'')$ is the average size of the cluster at step t'' , and L^d ($d = 2$) is the total number of spins of the system. t is proportional to the actual CPU time. This newly scaled time t should then be used in equation (15) for the exponential relaxation time of the Wolff dynamics.

The integrated correlation time should also be changed to

$$\tau_{\text{int}} = \sum_{t'=0}^{\infty} \frac{E(t') - E(\infty)}{E(0) - E(\infty)} \times \frac{C(t')}{L^d}. \quad (18)$$

In the Swendsen–Wang algorithm, there is no complication in the definition of time; we straightforwardly use equations (15) and (16) to calculate the dynamic exponent z_{exp} and z_{int} .

Our definition of the transformed time t is slightly different from that in [6]. In [6], the time t is defined by $t = t' C(t')$. If $C(t')$ takes a scaling form $C(t') \propto t'^{\alpha}$, the two definitions coincide. However, the disadvantage of the definition in [6] is that one needs to assume or know the scaling behaviour of the cluster size $C(t')$. Our definition of the time t has clear physical meaning; i.e., whenever the number of the flipped spins reaches L^d , it counts as a unit of time.

5. Results

We use the standard Hamiltonian of the two-dimensional Ising model,

$$-\beta H = K \sum_{\langle ij \rangle} \sigma_i \sigma_j. \quad (19)$$

Here $\beta = 1/(k_B T)$ and $K = J/(k_B T)$, k_B is the Boltzmann constant, T is the temperature, and J is the energy of interaction between two spins. Spins σ_i only take values $+1$ and -1 . The site i is on a square lattice with periodic boundary conditions.

Let us start our numerical simulations with the Swendsen–Wang algorithm. To calculate the dynamic exponent z_{exp} of the Swendsen–Wang algorithm, we have used lattices $L = 64, 128, 256, 512, 1024, 2048, 4096, 8192$, and each has $2^{24}, 2^{21}, 2^{19}, 2^{17}, 2^{15}, 5 \times 10^4, 4 \times 10^4, 2 \times 10^4$ runs respectively, and the maximum Monte Carlo time steps of each lattice are 60, 70, 70, 80, 90, 100, 100, 110. Here two different initial temperatures $T = \infty$ and 0 are used in the simulations. The system evolves at the critical temperature. From the exponential decay of the system energy in equation (15), one measures the relaxation time τ . The results are plotted on a linear–log scale in figure 2. The statistical errors in the figure are estimated by dividing the total of samples into two subgroups. The relaxation times are almost identical for both the hot start and the cold start. Obviously, an approximately linear behaviour is observed for the lattice size $L > 100$ in figure 2. In other words, a logarithmic dependence $\tau \sim \ln L$ gives a better fit to the numerical data. If we take into account the data for smaller lattices, the L dependence of the relaxation time could be given by $\tau \sim (\ln L)^{1.2}$.

If we analyse the data in the form of a power-law dependence, we find that the effective exponent z_{exp} decreases continuously with increasing lattice sizes, reaching 0.18 at the largest size simulated. This is shown with a log–log scale in figure 3, and it strongly suggests that the relaxation time does not follow a power law with a small but finite exponent. Our extensive data with large system sizes thus agree with the conclusion of Heermann and Burkitt [14]. Previous calculations [8], [15]–[19] gave various values ranging from 0.35 to 0.2. This appears to be an effect of finite lattice sizes.

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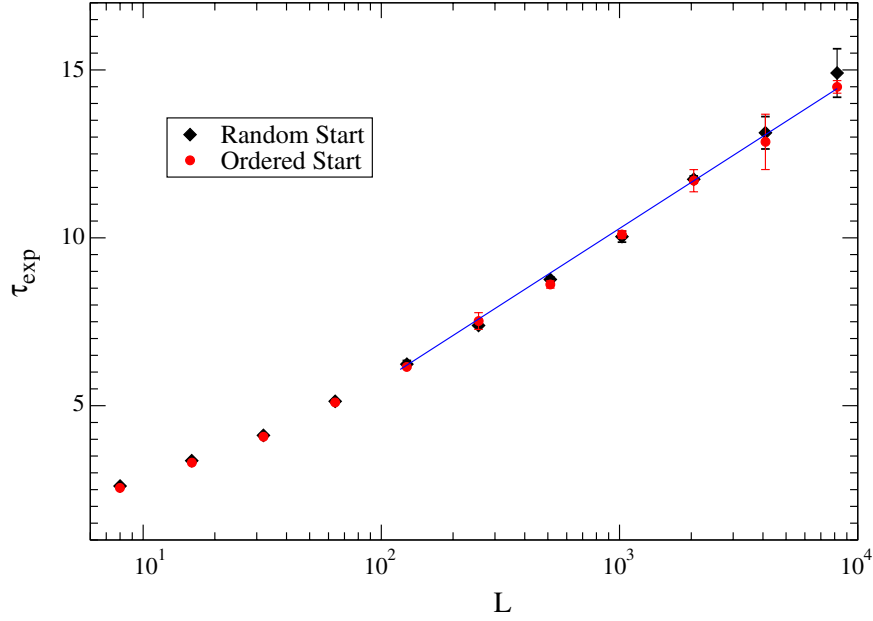


Figure 2. τ_{exp} versus L on a linear–log scale for the 2D Ising model with the Swendsen–Wang dynamics. The diamonds are for random initial configurations, and circles are for the ordered initial state. A straight-line fit gives $\tau_{\text{exp}} = -3.42 + 1.98 \ln L$.

For comparison, we also calculated the integrated relaxation time for the Swendsen–Wang dynamics; it is nearly a constant around $\tau_{\text{int}} \approx 3.1$. This implies that $z_{\text{int}} \approx 0$, which is consistent with z_{exp} .

We now investigate the Wolff dynamics with the fully ordered state at $T = 0$ as the initial state, and evolve the system at the critical temperature. The lattice sizes are $L = 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192$ with $2^{29}, 2^{27}, 2^{24}, 2^{21}, 2^{20}, 2^{20}, 2^{17}, 5 \times 10^5, 2 \times 10^5, 8 \times 10^4, 2 \times 10^4$ independent runs, respectively. The maximum Monte Carlo time steps for each lattice are 80, 100, 150, 180, 200, 200, 200, 220, 250, 250, 250. We observe that the dynamic behaviour here is very similar to that of the Swendsen–Wang dynamics. As shown in figure 4, the correlation time exhibits a logarithmic size dependence even from a relatively small lattice size. The integrated relaxation time is nearly a constant, $\tau_{\text{int}} \approx 1.17$ when the lattice size is larger than 256. This implies that $z_{\text{int}} \approx 0$. It is intuitively understandable that the Wolff dynamics starting from a fully ordered state is similar to the Swendsen–Wang dynamics. When the Swendsen–Wang algorithm is initialized in the order state ($T = 0$), and then evolves at critical temperature, it forms very large clusters which dominate the evolution of every Monte Carlo step, while the effect of other small clusters can be neglected. Thus its behaviour is much like a Wolff algorithm initializing at the ordered state, for it also has a very large cluster dominating the dynamic evolution. For the Swendsen–Wang algorithm with a disordered initial state, our simulations show that the clusters grow rapidly, and the relaxation time is almost the same as that with an ordered initial state.

For the Wolff dynamics with completely disordered initial states, we have used lattices $L = 64, 128, 256, 512, 1024, 2048$ with $2^{24}, 2^{23}, 2^{20}, 2^{17}, 2^{15}, 2^8$ runs respectively. The

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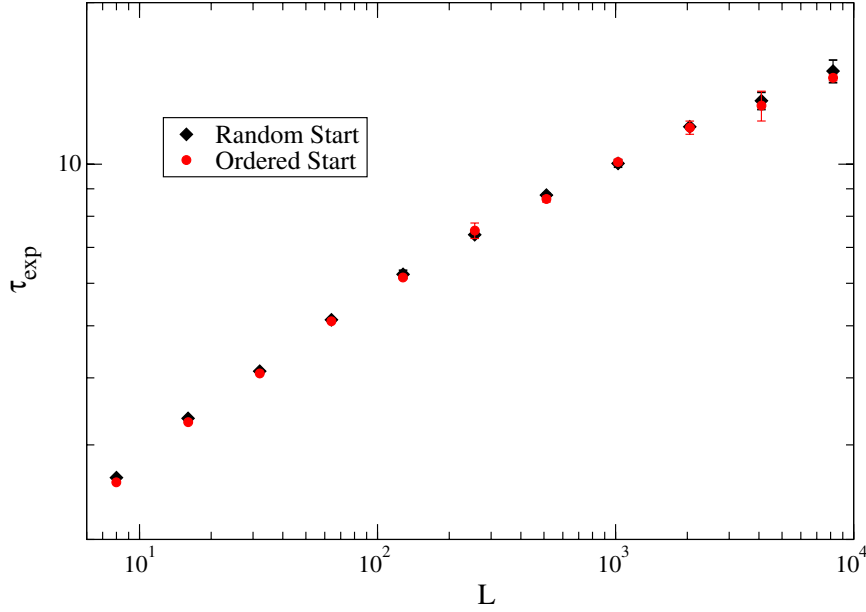


Figure 3. τ_{exp} versus L on a double-logarithmic scale for the 2D Ising model with the Swendsen–Wang dynamics.

maximum Monte Carlo time steps for each lattice are 450, 1337, 4770, 18 000, 69 910, 300 000 in the original time unit of t' . For the Wolff algorithm with a disordered initial state, the dynamic behaviour is rather complicated. In order to have a better comparison of the ordered and disordered initial starts, we present the dynamic relaxation of the system energy for the lattice size $L = 2048$ with two different initial conditions in figure 5. The figure shows that the dynamic relaxation with a disordered start is much slower.

In figure 6, τ_{exp} and τ_{int} are plotted as functions of the lattice size L on a log–log scale. The statistical errors are estimated by dividing the total of samples into two subgroups. For τ_{exp} , an approximate power-law behaviour is observed, and from large lattice sizes one derives a dynamic exponent $z_{\text{exp}} \approx 1.0$. To obtain a better value of z , one may consider corrections to scaling. For example, assuming $\tau_{\text{exp}} \sim L^z(1 + c/L^\delta)$, the fitted dynamic exponent is $z_{\text{exp}} = 1.19(2)$. Similarly we estimate $z_{\text{int}} = 0.29(3)$ from τ_{int} . z_{exp} is much larger than the results obtained with equilibrium correlation functions [15]. Both of these values are quite different from that reported in [6], where it is concluded that the exponent z is very close to zero.

To understand the difference between our analysis and that in [6], we have also performed the scaling plot of figure 1 in [6] with our numerical data. The scaling collapse is indeed observed for large lattices. According to our scaling analysis, however, a dynamic exponent $z \approx 1.7$ should be extracted. Following the definition in [6], $t = t' \langle C(t') \rangle / L^d$ with t' being the Monte Carlo time step of the Wolff single-cluster flip, and then $\tau = \tau' \langle C \rangle / L^d$. Assuming that $\langle C \rangle$ behaves like the susceptibility, i.e., $\langle C \rangle \sim L^{\gamma/\nu}$, one may deduce $z = z' - (d - \gamma/\nu) = z' + 2(Y_H - d)$. In [6], this is written as $z = z' - (2Y_H - d)$. Together with other inconsistent formulations, a dynamic exponent $z \sim 0$ is derived.

In numerical simulations in equilibrium, one tends to conclude that the dynamic exponent z of the Wolff algorithm is close to zero. This is in agreement with our results

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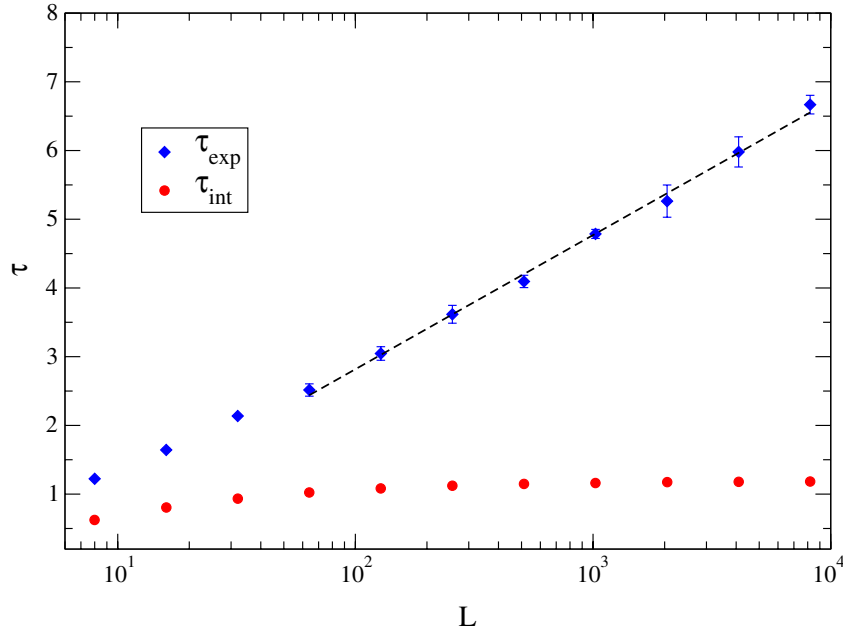


Figure 4. τ versus L on a linear–log scale for the 2D Ising model with the Wolff dynamics starting from an ordered initial state. The diamonds are exponential relaxation times, and a straight-line fit gives $\tau_{\text{exp}} = 1.09 + 0.85 \ln L$. The circles are integrated relaxation times. They reach a steady value near 1.17 for lattice sizes larger than 256.

from the dynamic relaxation starting from an ordered initial state. Compared with the dynamic relaxation of the Wolff algorithm with an ordered initial state, why does the dynamic relaxation with a disordered initial state show an anomalous behaviour? Our conjecture is that the eigenvalues λ_m of the transition matrix of the Wolff algorithm are rather dense, or at least quite different from that of the Swendsen–Wang case. Over a rather long time, the contribution of the higher eigenvalues to the dynamic observable will not be suppressed in the dynamic relaxation starting from a disordered state. The dynamic exponent measured in this paper and in [6] is actually an effective one. This also explains why the dynamic exponent extracted from the exponential decay of the system energy is somewhat smaller than that from the dynamic scaling behaviour in the relatively short time regime in [6].

Actually, a decade ago, Colonna-Romano *et al* [20] already investigated the dynamic relaxation to equilibrium for the 2D Ising model with the Wolff algorithm. The effective dynamic exponent was observed to be initial conditions dependent, for the system sizes and time windows they investigated. An estimated value $z = 1.1$ was reported. Our numerical simulations and data analysis also support that the effective dynamic exponent of the Wolff algorithm may depend on the initial states. If possible, one should avoid starting the simulations from a disordered state with the Wolff algorithm.

Here we should mention that for the dynamic relaxation of the Wolff algorithm with a disordered initial state, it is hard to measure τ_{exp} in an extremely long time regime where the contribution of the higher eigenvalues is really suppressed, since $|E(t) - E(\infty)|$

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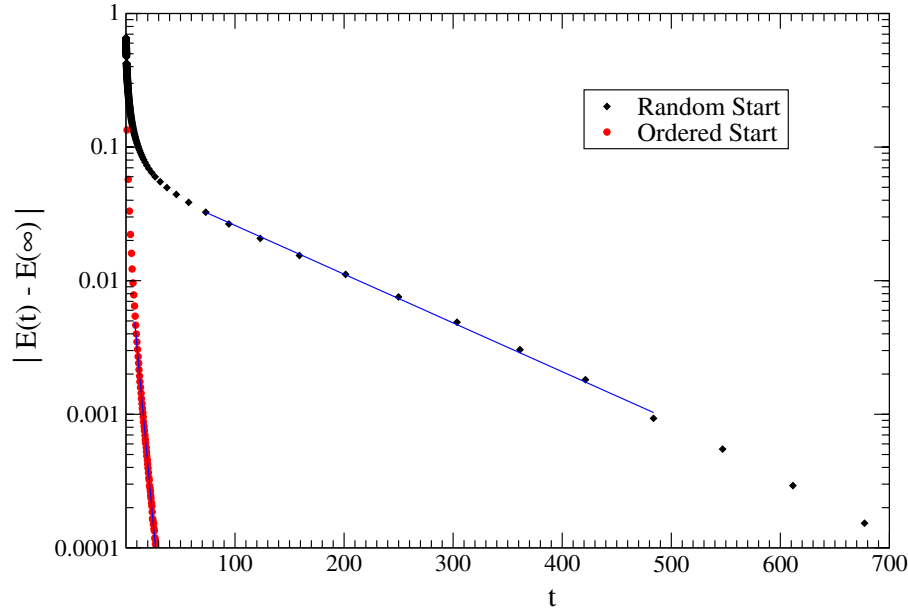


Figure 5. $|E(t) - E(\infty)|$ versus t on a log–linear scale for the 2D Ising model with transformed t for the Wolff dynamics. The lattice size is $L = 2048$.

Table 1. Correlation times and time windows for the measurements. The error in the column $\delta\tau$ is estimated by shrinking the time window to the first 2/3 and the last 2/3. Statistical errors are estimated by dividing the total samples into two subgroups.

Lattice	Time window	τ	$\delta\tau$	Statistical $\delta\tau$
8	[3, 14.5]	1.37	0.01	0.02
16	[10, 20]	1.99	0.03	0.01
32	[20, 30]	2.99	0.09	0.14
64	[27, 48]	4.80	0.31	0.10
128	[56, 78]	7.75	0.23	0.21
256	[90, 130]	13.36	0.87	0.33
512	[145, 200]	26.00	0.91	2.37
1024	[220, 305]	53.01	2.70	2.16
2048	[400, 750]	105.09	0.93	2.64

is too much fluctuating. In order to obtain a relatively reliable τ_{exp} for the Wolff algorithm with the disordered initial state, we have taken the time windows for the measurements to be the maximum linear part in the semi-log plots. In table 1, details of this kind are given. The error in the column $\delta\tau$ is estimated by shrinking the time window to the first 2/3 and the last 2/3. We observed that the errors from different time windows and statistical errors are comparable. In this sense, the curves presented in figure 6 should be reliable. For the Wolff algorithm starting from the ordered state and the Swendsen–Wang algorithm, the semi-log plots of $|E(t) - E(\infty)|$ show a relatively clean linear behaviour, and hence it is not difficult to estimate τ_{exp} . The results are quite robust against the choices of the fitting windows.

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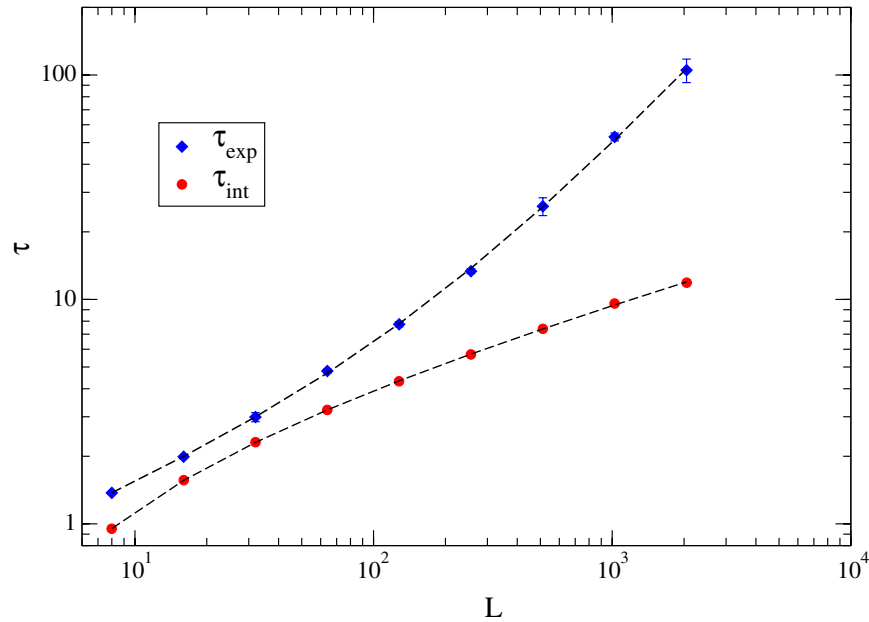


Figure 6. τ versus L on a double-logarithmic scale for the 2D Ising model with the Wolff algorithm starting from disordered states. The diamonds are exponential relaxation times τ_{exp} . The circles are integrated relaxation times.

6. Conclusion

In summary, we have computed both the exponential and integrated relaxation times for the Wolff and Swendsen–Wang algorithms, with both disordered and ordered initial states. For the Swendsen–Wang dynamics, the exponential relaxation time shows a logarithmic dependence on the lattice size L for both initial states, while the integrated relaxation time tends to a constant. This is strong evidence that $z_{\text{exp}} = 0$ for the Swendsen–Wang algorithm. For the Wolff dynamics with an ordered initial state, the results are similar to those for the Swendsen–Wang dynamics. For the Wolff dynamics with a disordered initial state, however, the dynamic relaxation is very slow, and it takes a very long time to approach equilibrium. If one measures the relaxation time τ_{exp} in a reasonable time regime in the simulations, an effective dynamic exponent $z_{\text{exp}} = 1.19(2)$ is obtained.

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