THE CRITICAL 2D ISING MODEL IN A MAGNETIC FIELD. A MONTE CARLO STUDY USING A SWENDSEN–WANG ALGORITHM

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We determine numerically the spin–spin correlation function in the scaling limit These data are useful in order to check regularization procedures à la Dotsenko, based on conformal theory, of perturbation series expansions

The local Monte Carlo simulation algorithms like Metropolis or heat-bath are not suited for studying properties near the critical point of a second-order phase transition because of critical slowing down Recently non-local algorithms have been developed for a whole series of models that either reduce critical slowing down considerably or even seem to eliminate it completely [1–8] In this paper we generalize the stochastic cluster algorithm that was developed by Swendsen and Wang [1] for the ferromagnetic Ising model without a magnetic field to the case of the critical Ising model in the presence of a magnetic field H We thus consider the partition function

$$Z=\sum_{S_{i,j}=\pm1}\exp\left(\beta_c\sum_{i,j=1}^{L}(S_{i,j}S_{i+1,j}+S_{i,j}S_{i,j+1}+HS_{i,j})\right), \tag{1}$$

where $\beta_c=\frac{1}{2}\ln(\sqrt{2}+1)$ We take periodic boundary conditions and compute the spin–spin correlation function

$$G^\sigma(R,H)=\langle S_{i,j}S_{i,j+R}\rangle-\langle S_{i,j}\rangle^2\,. \tag{2}$$

The algorithm is a straightforward extension of the Swendsen–Wang method [1]. The clusters are built in exactly the same way as in the $H=0$ case. Once the clusters have been found, each of the clusters is given an independent heat-bath update in the magnetic field. The probability that a cluster of N spins will be in the state $(+1)$ respectively (-1) are

$$P_+(N)=\frac{\exp(2\beta_c NH)}{1+\exp(2\beta_c NH)}\,,$$

$$P_-(N)=\frac{1}{1+\exp(2\beta_c NH)} \tag{3}$$

Although slightly more complicated than in the $H=0$ case it is also possible to use improved estimators [8] for $H\neq0$ This will give an additional gain by reducing the statistical noise of the measurements The improved estimators for more complicated objects like the energy–energy correlation function have been derived and will be discussed elsewhere [9]. For the 3D Ising model a similar extension of the algorithm has been considered [10] together with an alternative method where a ghost-spin is introduced

We now present our results We first consider the *large-R* (*fixed-H*) behavior In this case one expects according to Zamolodchikov [11]

$$G^\sigma(R,H)\simeq\sum_{i=1}^{8}a_i^\sigma(H)[K_0(m_iR)+K_0(m_i(L-R))] \tag{4}$$

The second term in eq. (4) was introduced in order to take into account the periodic boundary conditions. The ratio of the eight masses $m_i(H)$ are known:

$$m_2(H)=2m_1(H)\cos(\tfrac{1}{5}\pi)\,,$$

$$m_3(H)=2m_1(H)\cos(\tfrac{1}{30}\pi)\,,\quad \text{etc}\,, \tag{5}$$

but the coefficient functions a_i are not known Henkel and Saleur [12] have studied numerically the spectrum of the one-dimensional quantum Ising chain and confirmed the predictions of Zamolodchikov concerning the mass ratios, a similar check was done by von Gehlen [13] for the tricritical Ising model We would like to stress that these tests do not determine the coefficients a_i^σ A result of our own measurement on $G^\sigma(R, H)$ show that one is compatible with only one term in eq (4) The measured values of the correlation length $\xi = 1/m_1$ are shown in table 1 The fits with the function $K_0(R_0/\xi)$ for $H=0\,001$ and $H=0\,15$ were less good In the former case probably because of finite-size effects and in the latter case because the correlation length is too small (recall that for large values of z, $K_0(z) \simeq \sqrt{\pi/2z}\,e^{-z}$) A fit to the data of the form

$$\xi = AH^{-y} \tag{6}$$

gives $y=0\,55(2)$ and $A=0.36(1)$, in good agreement with the expected value $y=\frac{8}{15}$ If one fixes y at its theoretical value one finds $A=0.38(1)$. For the coefficient a_1^σ a fit of the form

$$a_1^\sigma = B^\sigma H^{z^\sigma} \tag{7}$$

gives $B^\sigma = 0\,144(4)$ and $z^\sigma = 0\,135(2)$, in agreement with the expected value $z^\sigma = \frac{2}{15}$ These values can be derived from elementary scaling arguments We consider now the scaling variables [14,15]

$$t = (R/\xi)^{15/8} \simeq 6\,1\,HR^{15/8} \tag{8}$$

Using eqs (4), (6) and (8) one can derive the following expressions for the correlation function for large values of R (or alternatively of t).

Table 1
The correlation length ξ for various values of the magnetic field H The values in parentheses indicate the error in the last digit For each value of H the measurements were made on an L^2 lattice with periodic boundary conditions The corresponding number of Monte Carlo update sweeps is also given

H	ξ	L	# sweeps ($\times 10^6$)
0 001	17 8(7)	128	0 15
0 0075	5 4(2)	64	5 5
0 02	3 12(3)	32	20
0 05	1 84(4)	32	7
0 1	1 26(5)	24	4 5
0 15	1 05(5)	24	4 5

$$G^\sigma(R, t) \simeq \frac{C^\sigma t^{2/15}}{R^{1/4}} K_0(t^{8/15}), \tag{9}$$

where $C^\sigma = 0\,113(4)$.

We consider now another limit of the correlation function [14,15] which is the scaling limit In this case one looks at the *large-R* (*fixed-t*) behavior In this case one expects

$$G^\sigma(R, t) = \frac{F^\sigma(t)}{R^{1/4}}, \tag{10}$$

where $F^\sigma(0)$ is known exactly [16] ($F^\sigma(0) = 0\,645002$) The function $F^\sigma(t)$ is of special interest because it can in principle be derived using perturbation theory starting from the conformal invariant point $t=0$ Unfortunately the regularization procedure is not obvious (it probably implies two free parameters) and thus the knowledge of F^σ would be a check of the procedure The large-t behavior of F^σ is given by eq (9) In fig. 1 we show the estimates for the function $F^\sigma(t)$ using five values of H Since $F^\sigma(t)$ is strongly dependent on t even for $t<1$ and this is the surprise coming from our work, we have separated the range of t in two separate intervals As one notices from the figure, up to $t \simeq 5$ the data scale very nicely and they do so also for larger values of t if one omits the values corresponding to $H=0\,1$ where the correlation length is presumably too small and $H=0\,001$ because of finite-size effects

We did not attempt to compare our data with Dotsenko's small-t expansion [16] (five terms are known, they contain powers of t and $\log(t)$) since we feel that at the present stage, this comparison should be left to the people who are deriving theoretically the function F^σ, especially so since it is not clear which is the convergence radius of the expansion To illustrate the problem, one gets a good approximation to $F^\sigma(t)$ for $t>0\,05$ taking

$$F^\sigma \simeq 0\,141 \frac{e^{-t^{8/15}}}{t^{2/15}}, \tag{11}$$

which can be derived from the large-t behavior given by eq (9) (see fig. 1a) and it is possible that in order to check the perturbation theory one has to go to values of t smaller than 10^{-4} For this reason we are going to make the raw data available to the interested reader [17] An open problem to which we hope to return

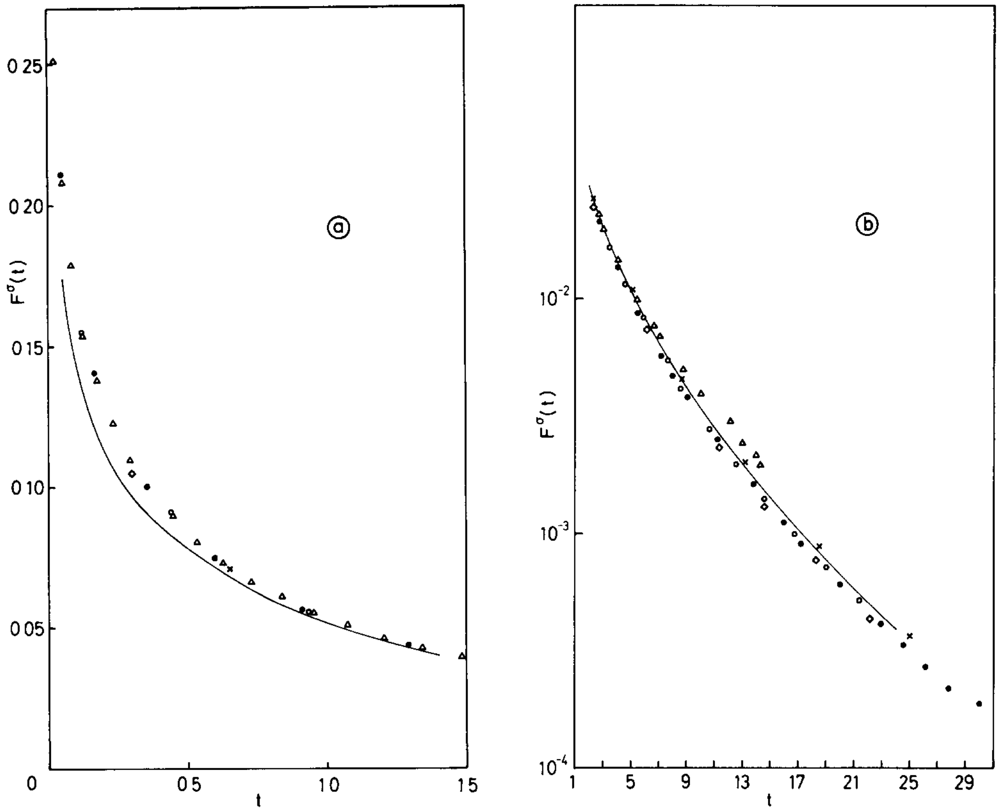

Fig 1 The $F^{\sigma}(t)$ function for various values of t (a) corresponds to the range $0<t<1$ 5 and (b) 1 $5<t<30$ ($\times$) corresponds to $H=0$ 1, ($\Diamond$) corresponds to $H=0$ 05, ($\circ$) corresponds to $H=0$ 02, ($\bullet$) corresponds to $H=0$ 0075, and ($\triangle$) corresponds to $H=0$ 001 The smooth curve corresponds to eq (11) Typical errors are less than 10^{-3} for $t<1$, less than 10^{-2} for $t<1$, and less than 5×10^{-2} for $t<30$

in the future is the universality of the function $F^{\sigma}(t)$

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