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CLUSTER MONTE CARLO ALGORITHMS

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Minireview

The Swendsen–Wang and Wolff Monte Carlo algorithms are described in some detail, using the Potts model as an example. Various generalizations are then reviewed and some applications are discussed. Two complete Fortran programs for the algorithms are provided.

1. Introduction

Monte Carlo (MC) simulation studies of critical phenomena [1] have been greatly hampered by critical slowing down [2]. As the critical point is approached, the correlation time, τ, diverges as $\tau \sim \xi^z$, where ξ is the spatial correlation length and z is called the dynamic critical exponent. For the finite systems used in computer simulations the correlation does not diverge, but it does increase with the system size as $\tau \sim L^z$, where L is the linear dimension of the system. For most MC algorithms, $z \approx 2$, and τ grows rapidly for large systems.

Since configurations are not statistically independent within a time interval of order τ, this greatly increases the statistical errors for observables. Alleviating or eliminating this problem is of great importance for high-accuracy computations.

The reason for critical slowing down is due to the fact that conventional Metropolis algorithms are local. Changes are propagated diffusively. It has long been recognized that non-local updating is necessary for a faster dynamics [3]. A number of specific methods have been proposed: Fourier acceleration [4], the multigrid Monte Carlo [5], algorithms based on renormalization group ideas [6], and over-relaxation algorithms [7].

Three years ago Swendsen and Wang (SW) [8] introduced an algorithm based on a mapping between the Potts model and a percolation problem [9]. Subsequently, advances have been made for other models [10–20]. This type of algorithms (now known as cluster algorithms) has been shown to be extremely effective in reducing critical slowing down. In this article we give an introduction to these algorithms and review the current status of various generalizations (see also refs. [21, 22] for reviews and [23] for a pedagogical introduction). We also discuss briefly the kind of problems that have been studied using these cluster algrothims. In the appendix, we provide two complete Fortran programs.

2. Cluster Algorithms

2.1. *Swendsen–Wang algorithm*

About two decades ago, Fortuin and Kasteleyn [9] found a relation between the Potts model and a percolation problem [24]. This made it possible to describe thermodynamic phase transitions by geometric percolation transitions [25, 26] and made Fisher's droplet picture [27] rigorous. Sweeny devised an algorithm to simulate the Potts model in the percolation representation in two dimensions [28]. It was realized only recently that both the spins and percolating clusters can be used to construct a MC algorithm [8].

We follow both Swendsen and Wang [8] and Edwards and Sokal [12] (see also ref. [29]) in describing the algorithm. Let us consider a Potts model defined by the probability distribution

$$P(\sigma) = \frac{1}{Z} \exp\Big(K \sum_{\langle i,j \rangle} (\delta_{\sigma_i,\sigma_j} - 1) \Big) , \tag{1}$$

where K $(=J/k_{\mathrm{B}}T)$ is the coupling strength; $\sigma_i = 1, 2, \ldots, q$; the summation runs over nearest neighbour pairs; Z is the partition function.

A SW Monte Carlo move consists of two steps: the first step transforms a Potts configuration to a bond configuration; the second transforms back from bond to a new Potts configuration.

Step 1: Create a bond, $n_{ij} = 1$, between neighbour site i and j stochastically with a probability $p = 1 - \mathrm{e}^{-K}$, if $\sigma_i = \sigma_j$. No bond will be present otherwise, the bond variable sets to $n_{ij} = 0$.

Step 2: Identify clusters as sets of sites connected by bonds, or isolated sites. Two sites are said to be in the same cluster if there is a connected path of bonds joining them. Each cluster is assigned a new Potts value chosen with

equal probability among 1 to q. The Potts variable σ' now takes the value of the cluster it belongs to.

The algorithm is highly ergodic, since every state can be reached from any other state in one move with a non-zero probability. In order to demonstrate that the algorithm generates an equilibrium distribution, eq. (1), it is sufficient to show that steps 1 and 2 leave the probability distribution invariant. The reader who believes step 1 and 2 to be correct may skip the remainder of this subsection.

In fig. 1 we consider a case of a 4×4 lattice (with $q = 2$, identifying $\sigma_i = 1, 2$ with $+, -$). Spins are denoted by plus and minus signs, and bond variables are shown by solid lines denoting $n_{ij} = 1$. Fig. 1a shows a typical Potts configuration. Assuming that a configuration appears with the probability given by eq. (1), we associate a factor 1 for $\sigma_i = \sigma_j$, and e^{-K} otherwise. The total weight or probability is the product of all factors. Step 1 amounts to creating a bond configuration for given σ with the (conditional) probability

$$P(n|\sigma) = \prod_{\langle i,j \rangle} \{[p\delta_{n_{ij},1} + (1-p)\delta_{n_{ij},0}]\,\delta_{\sigma_i,\sigma_j} + \delta_{n_{ij},0}[1 - \delta_{\sigma_i,\sigma_j}]\}\,, \tag{2}$$

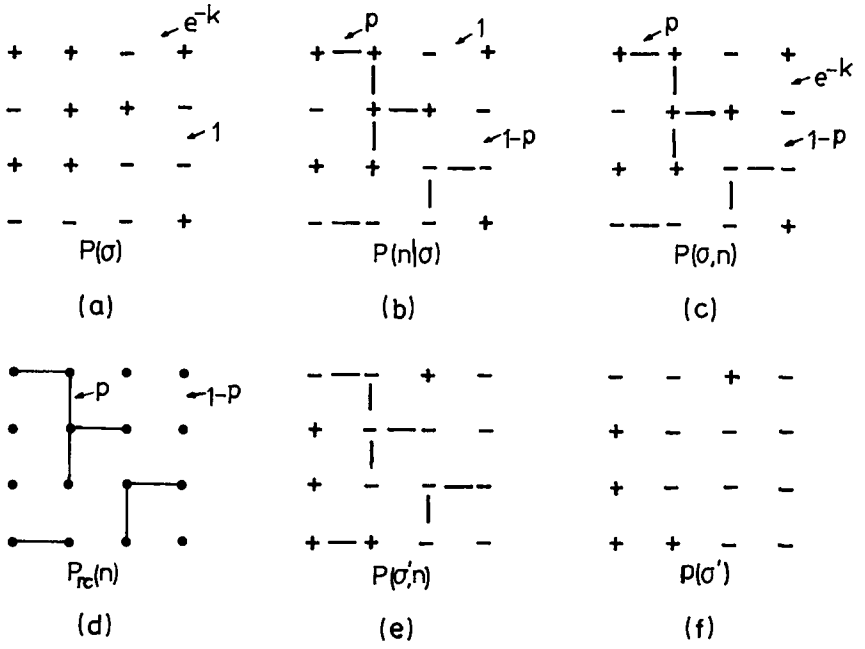

Fig. 1. Various steps of a SW Monte Carlo move: (a) a Potts state; (b) the conditional probability for bonds given the spins; (c) a configuration of the joint model of Potts spins and bonds; (d) clusters; (e) a new state for the joint model; (f) a new Potts state.

which is graphically represented by fig. 1b. Considering both σ and n as dynamical variables, we arrive at a site–bond correlated percolation problem, with a joint probability distribution

$$P_{\text{joint}}(\sigma, n) = P(n|\sigma)\, P(\sigma)\,. \tag{3}$$

By construction, summing over the bond configurations we get back to the original Potts model, eq. (1), since the transition, eq. (2), is normalized. If we ignore the Potts spins and look only at the bonds, they are distributed according to

$$P_{\text{rc}}(n) = \sum_{\sigma} P_{\text{joint}}(\sigma, n) = \frac{1}{Z}\, p^{b}(1-p)^{dL^d - b} q^{N_c}\,, \tag{4}$$

where b $(=\Sigma_{\langle i,j\rangle}\, n_{ij})$ is the number of bonds, $dL^d - b$ the number of absent bonds (assuming a d-dimensional hypercubic lattice of size L), N_c is the number of clusters of the configuration n. This is just the Fortuin–Kasteleyn mapping [9] of the Potts model to the percolation model (fig. 1d random cluster model).

For the joint model of Potts spins and bonds (fig. 1c), we note that for each bond one associates a weight p. There are two different situations when a bond is missing: either due to $\sigma_i \neq \sigma_j$, with weight e^{-K}, or due to the bond probability $1-p$. The choice $\mathrm{e}^{-K} = 1-p$ means that there are no interactions between clusters. Thus the state of each cluster can be changed independently, which is the Monte Carlo move of step 2. Step 2 can also be viewed as creating a Potts state given the bond configuration n,

$$P(\sigma'|n) = q^{-N_c} \prod_{\langle i,j\rangle : n_{ij}=1} \delta_{\sigma_i, \sigma_j'}\,. \tag{5}$$

The product of Kronecker δ's ensures the consistency of Potts states with the bonds. This transition probability clearly gives the joint probability back (fig. 1e). The new Potts spin is then distributed according to

$$P'(\sigma') = \sum_{n} P(\sigma'|n)\, P_{\text{rc}}(n) = \sum_{n} P_{\text{joint}}(\sigma', n) = P(\sigma')\,. \tag{6}$$

We have seen that the probability distribution, eq. (1), is unchanged under the transition

$$W(\sigma \to \sigma') = \sum_{n} P(\sigma'|n)\, P(n|\sigma)\,. \tag{7}$$

It is straightforward to verify that a stronger condition, detailed balance,

$$P(\sigma)\, W(\sigma \to \sigma') = P(\sigma')\, W(\sigma' \to \sigma) \tag{8}$$

is satisfied.

Of course, an efficient implementation of the algorithm relies on the ability to perform step 2 by a computer in operations of order L^d. Such a cluster labelling algorithm has been developed for percolation problems [30]. A discussion of the algorithm is given in the appendix.

2.2. *Wolff's algorithm*

Wolff [15] has successfully generalized SW algorithm in two distinct ways: (1) the single cluster method; (2) generalization to O(n) models. Here we will discuss the single-cluster method for Potts models, and treat the generalization to O(n) models later in the paper.

A single cluster is grown from a seed very much the same as that of Leath [31] for generating clusters in a percolation problem. A starting site is chosen at random. The neighbours of the chosen center site are included as a member of the cluster with the same probability as that in SW,

$$p(\sigma_i, \sigma_j) = 1 - \exp(-K\delta_{\sigma_i,\sigma_j})\,. \tag{9}$$

New neighbours are considered if that particular pair were not visited before. This process continues until no more new sites are generated. For $q = 2$ the spins of the cluster are flipped. If $q > 2$, one chooses a new state with equal probability among the q states.

Detailed balance is satisfied by noting that the transition probability from σ to σ' via a particular cluster is

$$W(\sigma \to \sigma', \mathrm{C}) = w_{\mathrm{bulk}}(\sigma, \mathrm{C}) \prod_{\langle i,j\rangle \in \partial \mathrm{C}} [1 - p(\sigma_i, \sigma_j)]\,, \tag{10}$$

where C denotes the cluster; ∂C is the surface of C consisting of bonds connecting the sites in the cluster and sites outside the cluster. Clearly, the perimeter must not contain a bond. For the reverse transition probability $W(\sigma' \to \sigma, \mathrm{C})$, the bulk part is the same due to the symmetry of the model, and only the surface part differs. This difference provides exactly the desired form for detailed balance.

The Wolff single-cluster algorithm is more efficient than the SW algorithm in higher dimensions [32, 33].

2.3. Variance reduction

Cluster algorithms offer additional advantage for calculating thermodynamical averages. The clusters themselves contain useful information [26, 28]. For example, the susceptibility is given by the second moment of the cluster number distribution, and specific heat is related to the fluctuation of the number of bonds. These relations can be readily derived from the joint distribution [29, 34].

Since one cluster configuration corresponds to many spin configurations, the variance is smaller and the results are more accurate if one calculates quantities from the former. This aspect has been emphasized recently by a number of authors [10, 17, 35].

3. Generalizations

There have been several attempts to generalize the SW algorithm to continuous spin models. In 1988, Wolff [10] simulated an annealed, bond-diluted version of the O(3) model with a non-local move of random rotation of each cluster. The result was negative: critical slowing down was not reduced.

Niedermayer [11] proposed a general scheme for an arbitrary model. One generates clusters by creating a bond at each interaction link with a probability $p_l(E_l)$ depending on the energy E_l. The functional form is arbitrary and is chosen to minimize correlations. For general $p_l(E_l)$, the clusters will interact through a surface energy defined by

$$e^{-E_{\text{surface}}} = \prod_{l \in \partial C} e^{-E_l}(1 - p_l) , \tag{11}$$

where the product runs over all links of the surface of the cluster C; a factor $1/k_B T$ has been absorbed into the definition of energies. The state of the cluster is updated by a move of the global symmetry of the model, which is accepted or rejected according to the change in surface energy, in analogy to the Metropolis algorithm. Note that the particular choice p of SW for the Potts model gives $\delta E_{\text{surface}} = 0$. Thus clusters take new randomly chosen spins. The performance for the XY model using random rotation was not great [11, 16]. In a study of the O(4) model in four dimensions the algorithm is shown to be worse than a standard Metropolis updating [36].

Edwards and Sokal [12] gave a generalization which emphasizes the role of the two conditional probabilities. An augmented model is introduced where variables associated with bonds take real values. The difficulty encountered

here is that the conditional probability for the original configuration given the bond variables is not, in general, simple. Thus it is not possible to generate an independent distribution, and it is necessary to resort to other Monte Carlo methods. The generalized SW and multigrid algorithms [5] behave similarly for the XY model: there is still critical slowing down on the high-temperature side of the phase transition, but not on the low-temperature side.

Kandel et al. [13, 14] combined the idea of multigrid and a stochastic blocking procedure, giving a correlation time $\tau \approx 3$ for sizes $8 \leq L \leq 128$ for the 2D Ising model. The dynamic critical exponent z was claimed to be zero. In this approach, instead of transforming all the interactions in one step, the bonds are built up step by step in a hierarchical fashion. A sequence of coarsening models with less degrees of freedom are constructed and simulated, with a process of uncoarsening, backing to the original, finest level. The algorithm is very promising, although a practical implementation requires programming skill.

The most successful generalization of cluster algorithms to continuous spin models is that of Wolff [15, 16]. His essential idea was to embed an Ising variable in an O(n) model as a reflection of each spin through a plane, which is randomly chosen each time, but identical for all spins in the lattice. Ising embeddings have also been carried out for the O(n) model [17] with a slightly different algorithm, the ϕ^4 model [18], and anti-ferromagnetic Potts model [19].

In the ϕ^4 model, the real variable ϕ is decomposed into a magnitude and a sign. The sign represents an Ising variable and is updated using either the SW algorithm or Wolff's algorithm.

For the anti-ferromagnetic Potts model [19], one chooses at random two pairs of Potts states and attempts to interchange them. Again the operation can be viewed as an embedding of Ising variables, and can be considered as simulating a diluted anti-ferromagnetic Ising model with either the single cluster or the SW algorithm. For the $D = 2$ three-state anti-ferromagnetic Potts model, the dynamic critical exponent (at $T = 0$) is very small, and appears to be zero. For the $D = 3$ three-state anti-ferromagnetic Potts model, the dynamic critical exponent (at $T = T_c$) is about 0.5, as opposed to 2 for standard MC simulations.

Important progress has been made for a $\mathbb{Z}_2$ lattice gauge theory in three dimensions [20]. A rather ingenious algorithm was found to transform back from a deleting/freezing plaquette configuration (analog of a bond configuration in the Potts model) to a new field configuration. The dynamical behavior is believed to be in the same universality class of the SW Ising model.

Systems with competing interactions, spin glasses, and (continuous) lattice gauge fields have not been treated efficiently with cluster algorithms.

4. Dynamics

Time correlations can be measured by the (equilibrium) correlation function

$$f(t) = \frac{\langle A(t'+t)\, A(t')\rangle - \langle A(t')\rangle^2}{\langle A(t')^2\rangle - \langle A(t')\rangle^2}\,, \tag{12}$$

where the average is over Monte Carlo steps and time t is measured in units of one updating per degree of freedom. $A(t')$ is some observable containing the slowest mode of the dynamics, usually chosen to be energy, magnetization, or square magnetization. A correlation time, τ_{exp}, is defined by

$$f(t) \approx a \exp(-t/\tau_{\text{exp}})\,, \qquad t \text{ large}\,, \tag{13}$$

or by a sum

$$\tau_{\text{int}} = \sum_{t=0}^{\infty} f(t)\,. \tag{14}$$

As emphasized by Sokal [21], the exponential correlation time is an intrinsic property of the dynamics, while the integral correlation time is relevant to statistical error.

The dynamic critical exponent is most easily extracted through a finite-size scaling at the critical point,

$$\tau_{\text{exp}}, \tau_{\text{int}} \sim L^z\,, \qquad L \to \infty\,. \tag{15}$$

The exponents for cluster algorithms have been obtained for a variety of models (see table I). The Metropolis algorithm gives $z \approx 2$ for order-parameter nonconserving dynamics not very sensitive to dimensionality and models. For cluster updating algorithms, different degrees of improvement from $z = 0$ to 1 have been achieved. For the same algorithm and model, different authors tend to get different results, reflecting the difficulty of determining the exponent. The reason may be that the correlation time τ is very small for the cluster algorithms and systematic error due to small L appears particularly large.

The SW dynamic critical exponents of the Ising model were at first estimated to be $z(2\text{D}) = 0.35$ and $z(3\text{D}) = 0.75$. Recent extensive calculations indicate that the exponents are in fact smaller [32, 39, 40]. In two dimensions Heermann and Burkitt [39] suggested a logarithmic divergence $\tau \sim \ln L$.

Critical slowing down is practically completely eliminated for a number of systems: 1D Ising, three-state anti-ferromagnetic Potts model and XY model in two dimensions, etc.

Table I
Dynamic critical exponents of cluster algorithms for various models.

Model	Dimension	Algorithm	z	References
Ising	1	SW	0 (exact)	[37, 18]
	2	SW	0.35 ± 0.01	[8]
			0.20–0.27	[32]
			0 (log)	[39]
		Wolff	0.13–0.26	[32]
			0.33 ± 0.05	[33]
		Kandel et al.	0	[13]
	3	SW	0.75 ± 0.01	[8]
			0.50 ± 0.03	[32]
			0.339–0.46	[40]
		Wolff	0.14–0.28	[32]
			0.44 ± 0.01	[33]
	∞	SW	0.98 ± 0.08	[42]
		Wolff	0	[33]
$q = 3$ Potts	2	SW	0.6 ± 0.1	[8]
			0.4	[14]
			0.55 ± 0.03	[43]
		Kandel et al.	<0.2	[14]
$q = 4$ Potts	2	SW	0.89 ± 0.05	[43]
$q = 3$ A-Potts	2	SW	0	[19]
	3		0.48 ± 0.04	[19]
XY	2	Wolff	0	[15, 38]
	3	Embedded SW	0.46 ± 0.03	[45]
O(3)	2	Wolff	0	[15]
Φ^4	1	Embedded SW	0.07 ± 0.07	[18]
	2		0.29 ± 0.09	[18]
	3		0.87 ± 0.20	[18]
$\mathbb{Z}_2$ gauge	3	Ben-Av et al.	0.73 ± 0.05	[20]

5. Theories

Our theoretical understanding of the non-local algorithms is still preliminary. Klein, Ray, and Tamayo [41] have proposed a scaling ansatz relating the SW dynamic critical exponent to that of Glauber dynamics. They assume that SW dynamics is still governed by domain-wall diffusion process as for the local algorithms, the difference is that the smallest length scale changes in SW. This leads to the following relation:

$$z_{sw} = z_G - 2\gamma/d_m\nu \,, \tag{16}$$

where the mean fractal dimension of finite clusters satisfies $d - \beta/\nu \leq d_m \leq d$. A comparison with old z_{sw} results [8] suggested $d_m \approx d$. The recent new values

indicate a smaller d_m [40] or even that the equation is not obeyed if $z_{sw}(2D) = 0$ [39].

Mean field calculations for SW dynamics as well as Wolff's single cluster variation give $z_{sw} = 1$ [42] and $z_w = 0$ [33] for the Ising models above the upper critical dimension four.

A rigorous lower bound,

$$z_{sw} \geq \alpha/\nu \,, \tag{17}$$

has been obtained by Li and Sokal [43], where α is the specific heat exponent, ν the correlation length exponent. This is very similar to that for Glauber dynamics ($z_G \geq \gamma/\nu$) with the slowest mode replaced by the energy fluctuation.

6. Applications

With the new algorithms at our disposal, many high precision studies are now possible. In a study of the three-state anti-ferromagnetic Potts model [19], very accurate exponents in two dimensions were obtained, and later confirmed by analytic calculations [44]. In three dimensions, the critical behaviour is found to be consistent with that of the 3D XY model [19]. This is supported by a study of the 3D XY model using a cluster update method [45]. A high precision study of the 2D XY model has been carried out [16], supporting a Kosterlitz–Thouless type critical behaviour.

The SW and Wolff algorithms have been used to study dilute Ising models. In two dimensions [46], the study has clarified two controversial analytical theories. The results are consistent with a doubly logarithmic divergence for the specific heat and 2D Ising exponents with logarithmic corrections for the magnetization and susceptibility. Similar work in three dimensions gives some indications for universality [47].

The new algorithms also stimulated several studies of the cluster size distributions of the Ising model without a magnetic field [29] and in a magnetic field [48–50] (see also ref. [51]). The radius of gyration of the clusters in the 3D Ising at the critical point has been calculated and the result is consistent with a fractal dimension $d - \beta/\nu$ [52].

A few applications have gone off the equilibrium problems. Stauffer has considered a damage spreading in SW dynamics [53]. Domain growth by quench from a disordered state to a two-phase region has been studied by Burkitt and Heermann [54]. Metastability and nucleation in SW dynamics was initiated by Ray and Tamayo [55, 56].

Applications in field theory are outside the scope of this review.

7. Conclusion

Cluster algorithms have proven to be remarkably versatile and effective. Although our understanding of why they work so well, and what their limitations are, is still very rudimentary, they have opened up new possibilities for computer simulations in statistical mechanics. Work is in progress in several areas both to exploit current advances and to develop the new approach further.

Acknowledgement

The two programs in the appendix were due to a joint effort of J.-S. W. and D. Stauffer. We thank him for his insistence on short and efficient codes.

Appendix

In the following two programs for the 2D Ising model, we use helical boundary conditions. More precisely, the sites are designated by a single number from 1 to L^2. The four neighbours of site I are $I \pm 1$, $I \pm L$. If these numbers are outside the range, the spin IS is simply set to zero: namely a free boundary for two of the edges. A multiplicative (linear congruential) random number generator is used, generating numbers from -2^{31} to $2^{31} - 1$, which is the simplest but not one of high quality. For serious calculations one should use better random number generators. The first SW program takes 2.2 μsec on IBM 3090, 18 μsec on Vax 3200, per Monte Carlo step per spin. Wolff's algorithm has a comparable speed. Both of the programs are not vectorized.

In the first program, the labelling procedure is slightly different from that of Hoshen and Kopelman [30]. Here, each site has a pointer (LIST) pointing to a member in the currently identified cluster. The proper label is the smallest one pointing to itself. The labels are then renamed and sequenced. INC contains the total number of clusters. See ref. [24] for a detailed explanation.

For Wolff's algorithm, we flip the spin immediately once we know that it belongs to the cluster. This has the advantage that we need not keep track which site has already been visited before. The stack (ISTACK) contains these sites of the cluster whose neighbours are yet to be visited. The cluster growth terminates when the stack is empty, indicating by the pointer IPT = 0.

```
C     SW algorithm of 2D Ising model
      PARAMETER (L=100, L2=L*L)
```

```
      DIMENSION LIST(L2), IS(L2)
      DATA T, IBM, MCHALF/1.0, 1, 50/
      IPR = (1.0 - 2.0*EXP(-2.0/T)) * 2147483647.0
      MCSTEP = MCHALF * 2
      MAV = 0
      DO 1 I = 1, L2
1     IS(I) = 1
      DO 2 MC = 1, MCSTEP
      LIST(1) = 1
      DO 3 I = 2, L2
      IF (IS(I-1).NE.IS(I)) GOTO 4
      IBM = IBM * 16807
      IF (IBM.GT.IPR) GOTO 4
      LIST(I) = LIST(I-1)
      GOTO 5
4     LIST(I) = I
5     IF (I.LE.L) GOTO 3
      MB = I - L
      IF (IS(MB).NE.IS(I)) GOTO 3
      IBM = IBM * 65539
      IF (IBM.GE.IPR) GOTO 3
6     MB = LIST(MB)
      IF (MB.NE.LIST(MB)) GOTO 6
      MA = LIST(I)
      ISMALL = MINO(MA,MB)
      LIST(MAXO(MA,MB)) = ISMALL
      LIST(I) = ISMALL
3     CONTINUE
      INC = 0
      DO 7 I =1, L2
      IF (I.EQ.LIST(I)) THEN
        INC = INC + 1
        IS(I) = INC
      ELSE
        MA = LIST(I)
8       MA = LIST(MA)
        IF (MA.NE.LIST(MA)) GOTO 8
        IS(I) = IS(MA)
      END IF
7     CONTINUE
      DO 9 I = 1, INC
```

```
      IBM = IBM*16807
 9    LIST(I) = ISIGN(1,IBM)
      MAG = 0
      DO 10 I = 1, L2
      IS(I) = LIST(IS(I))
10    MAG = MAG + IS(I)
2     IF (MC.GT.MCHALF) MAV = MAV + IABS(MAG)
      PRINT *, MAV*2.0/(L2*MCSTEP)
      END


C     Wolff's algorithm of 2D Ising model
      PARAMETER (L=32, L2=L*L, MULT=16807)
      INTEGER IS(-L:L2+L), ISTACK(L2), NN(4)
      DATA T/1.0/,IBM/1/,MCDIS/100/,MCSTEP/100/,IPT/0/,X/0.0/
      IPR = (1.0-2.0*EXP(-2.0/T))*2147483648.0
      DO 2 I = -L, L2+L
      IS(I) = 1
 2    IF (I.LE.0 .OR. I.GT.L2) IS(I) = 0
      DO 1 MC = 1, MCSTEP + MCDIS
      ISIZE = 1
      IBM = IBM * MULT
      I = INT((IBM*2.3283E-10+0.5)*L2) + 1
      ISO = IS(I)
      IS(I) = -ISO
3     NN(1) = I + 1
      NN(2) = I + L
      NN(3) = I - 1
      NN(4) = I - L
      DO 4 J = 1, 4
      NNJ = NN(J)
      IF (ISO.NE.IS(NNJ)) GOTO 4
      IBM = IBM * MULT
      IF (IBM.GE.IPR) GOTO 4
      ISIZE = ISIZE + 1
      IS(NNJ) = - IS(NNJ)
      IPT = IPT + 1
      ISTACK(IPT) = NNJ
4     CONTINUE
      IF (IPT.LE.0) GOTO 1
      I = ISTACK(IPT)
```

```
      IPT = IPT - 1
      GOTO 3
1     IF (MC.GT.MCDIS) X = X + ISIZE
      PRINT *, X/MCSTEP
      END
```

Note added in proof

Vl.S. Dotsenko, W. Selke and A.L. Talapov (preprint) proposed cluster algorithms for random field and spin glass models. The efficiency of the algorithms remains to be seen.

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