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CRITICAL SLOWING DOWN

Ulli WOLFF

Fakultät für Physik, Universität Bielefeld, D-4800 Bielefeld, FRG
and
Institut für Theoretische Physik, Universität Kiel, D-2300 Kiel, FRG

The problem of critical slowing down in Monte Carlo simulations and some methods to alleviate or overcome it are reviewed: overrelaxation, multigrid and cluster algorithms.

1. INTRODUCTION

The standard problem making up a large part of statistical mechanics and lattice quantum field theory is (in the language of the latter) the evaluation of path integrals of the type

$$\langle A(\varphi)\rangle = \frac{1}{Z}\int \prod_{x\in\Lambda} d\varphi_x e^{-S(\varphi)} A(\varphi). \qquad (1)$$

Here φ_x is a field on a lattice Λ, S is the action and A some observable. In the Monte Carlo approach to (1) one generates a sequence of configurations $\varphi^{(i)}, i = 1,\ldots,N$, for which the observable assumes values $A^{(i)} = A(\varphi^{(i)})$, such that their mean is an estimate of (1),

$$\bar{A} = \langle A(\varphi)\rangle \simeq \frac{1}{N}\sum_{i=1}^{N} A^{(i)} \qquad (2)$$

The algorithm producing the $\{\varphi^{(i)}\}$ on a computer corresponds to a set of transition probabilities $W(\varphi^{(i)} \to \varphi^{(i+1)})$. It is well known that W has to be ergodic, and that the Boltzmann factor has to be stable under W,

$$\int \prod_{x\in\Lambda} d\varphi_x e^{-S(\varphi)} W(\varphi \to \varphi') = 1 \times e^{-S(\varphi')}. \qquad (3)$$

I emphasize that (3) may be read as an eigenvalue equation with an eigenvalue 1 and eigenvector $e^{-S(\varphi)}$. The properties of W as a probability imply [1,2] that all *other* eigenvalues λ obey

$$\sup|\lambda| = e^{-1/\tau} < 1. \qquad (4)$$

The so called exponential autocorrelation time τ parameterizes the gap between $\lambda = 1$ and the subleading (unwanted) modes. The property τ of W sets the scale for the number of configurations that have to be discarded for equilibration before $A^{(1)}$ is recorded. The expected error σ_A in (2) is given by

$$\sigma_A^2 = \langle(\frac{1}{N}\sum_{i=1}^{N} A^{(i)} - \bar{A})^2\rangle = \frac{1}{N^2}\sum_{i,j=1}^{N} \Gamma_A(i,j) \qquad (5)$$

with the autocorrelation function

$$\Gamma_A(i,j) = \langle(A^{(i)} - \bar{A})(A^{(j)} - \bar{A})\rangle$$

$$= \Gamma_A(i-j) \quad \text{(in equilibrium)}. \qquad (6)$$

If Γ_A decays in much less than N iterations, I derive

$$\sigma_A^2 \simeq \frac{\Gamma_A(0)}{(N/2\tau_{int,A})} \qquad (7)$$

with

$$\tau_{int,A} = \frac{1}{2}\sum_{t=-\infty}^{\infty} \Gamma(t)/\Gamma(0). \qquad (8)$$

The effective or integrated autocorrelation time $\tau_{int,A}$ sets the scale for the number of iterations with W required between statistically independent estimates of A. In short, it tells me how often to measure. In my experience it is both much easier and more significant to determine $\tau_{int,A}$ for a number of representative observables than to look for $\tau_{exp} = \tau$, if one just wants to evaluate (1) with good accuracy and is not primarily interested in the dynamics of the process. If the

τ_{int} are manageably small, equilibration can usually be achieved easily, too. The advantage of τ_{exp} is, of course, that it is a feature of W alone and thus theoretically nice. To estimate τ_{int} in practice from data with noise, it is important to truncate the sum in (8) at an appropriate time separation [2,26].

The continuum limit in (1) is approached through a critical point where a correlation length in lattice units ξ diverges. One speaks of critical slowing down if the autocorrelaion times blow up, too. This is typically governed by a dynamical scaling law

$$\tau_{(int,A)} \propto \xi^{z(A)}, \tag{9}$$

with dynamical critical exponents z and z_A. For standard local algorithms it is known that $z \simeq 2$, and (9) shows their inefficiency in the form of an increasing number of iterations that have to be carried out on ever larger systems in the critical limit. Improved algorithms aim at reducing z for updates of comparable complexity, i. e. o(volume) operations or "sweeps".

2. OVERRELAXATION

In 1981 Adler [3] showed that the technique of overrelaxation (OR)— known from algorithms to solve difference equations— can be generalized to the Monte Carlo simulation of multiquadratic systems. Their action depends on each individual degree of freedom φ_x only through the form

$$S = a(\varphi_x - F_x[\varphi_{y\neq x}])^2 + \varphi_x - \text{independent}. \tag{10}$$

A local update

$$\varphi_x \to \varphi'_x = \omega F_x + (1-\omega)\varphi_x + \sqrt{\frac{\omega(2-\omega)}{a}}\,\eta, \tag{11}$$

where η is a gaussian random number of unit variance, fulfills detailed balance for any $0 < \omega \leq 2$. Clearly, for $\omega = 1$ one performs a standard heatbath update, while for $\omega \to 0$ a discretized Langevin equation is approximated. For free fields it can be shown rigorously [4,5] that there is an optimal overrelaxed value

$$\omega = 2 - c/\xi \tag{12}$$

close to 2, such that $z = 1$ results. This improvement is reached essentially free of CPU-time cost, since one still performs local updates.

In its original form the restriction to multiquadratic systems limited the usefulness of OR. Generalizations to the compact variables mostly found in spin models and lattice gauge theory were suggested by Brown and Woch [6] and by Creutz [7]. Particularly easy is the analog of the deterministic microcanonical limit $\omega = 2$ of (11) where the energy is unchanged. For an $O(n)$ nonlinear σ-model, for instance, the standard action depends on an individual spin on the unit sphere, $s_x \in S_{n-1}$, through

$$S = \alpha s_x \cdot N_x[s_{y\neq x}] + s_x\text{−independent}, \tag{13}$$

with real scalar α. The direction $N_x \in S_{n-1}$ is given by the average over the neighboring spins. The update

$$s_x \to s'_x = -s_x + 2(s_x \cdot N_x)N_x \tag{14}$$

conserves (13). A way to verify detailed balance is: imagine (14) as a (symmetric) proposal for Metropolis updating, which has 100% acceptance. Being microcanonical, it is clear that (14) has to be combined with other conventional ergodic updates to simulate the canonical ensemble. This mixture may be interpreted as mimicking $\omega < 2$. This form of OR is easy to implement and has found several successful applications recently [8,9]. In [8] an exponent $z \approx 1.2$ is seen for the XY-model at least effectively for correlation lengths up to 22 on large lattices. This refers to a certain approximately optimized pattern of mixing whole OR-sweeps with Metropolis sweeps.

Very importantly, OR can also be adapted to $SU(n)$ gauge theory. Here the standard dependence on one link variable $U_{x\mu}$ looks like

$$S = ReTr(U_{x\mu}V^\dagger) + U_{x\mu}\text{−independent}, \tag{15}$$

where V is an $n \times n$ complex matrix. If S is maximal for $U_{x\mu} = W \in SU(n)$, the update proposal is defined by

$$U'_{x\mu} = WU_{x\mu}^{-1}W \tag{16}$$

For $SU(2)$ we have $W = \alpha V$ with a real scalar α, and **(16)** is microcanonical. For $SU(3)$ a nontrivial acceptance step has to be added. Typically, the acceptance comes out high for typical physical parameters, and **(16)** is almost microcanonical. Again, ergodic standard updates may be blended with **(16)**. Practical results for $SU(3)$ are found in [10]. The authors conclude that speed up factors of 2-3 in favor of OR in a comparison with Metropolis and pseudo-heatbath can be achieved. Their study used a 9^4 lattice and analyzed long range observables in the form of blocked Wilson loops. In a similar comparison Decker and de Forcrand (see contribution to these procedings) found a speed up of around 10 in CPU time for $SU(2)$ gauge theory on 32^4. In conclusion, it seems likely that some admixture of OR leads to the best presently available algorithm for pure gauge theory. One should however bear in mind, that such a statement may depend on the parameters and observables that are of interest.

3. MULTIGRID

The idea of a multigrid Monte Carlo (MGMC) algorithm—again borrowed from solvers of partial difference equations—was mentioned by Parisi [11] in 1983 and put to work by Goodman and Sokal [12] in 1986. The basic idea will be discussed for a scalar field φ_x which is first taken as one dimensional. Apart from the basic lattice Λ_b of 2^b sites, on which φ_x originally lives, we imagine a hierarchy of coarser lattices according to the scheme:

Any field φ on Λ_l with action $S_l(\varphi)$ may be updated in the standard fashion on Λ_l as well as by

$$\varphi \to \varphi + p_{l,l-1}\psi, \tag{17}$$

where ψ lives on Λ_{l-1} and $p_{l,l-1}$ is some linear interpolation operator. An effective action for the ψ-update is induced,

$$S_{l-1}(\psi) = S_l(\varphi + p_{l,l-1}\psi). \tag{18}$$

Starting from $\psi \equiv 0$, S_{l-1} is used to generate $\psi \not\equiv 0$ by some local Monte Carlo algorithm. The simplest choice for $p_{l,l-1}$ is "piecewise constant injection", where the pair of sites on Λ_l associated with (=closest to, see drawing) each site of Λ_{l-1} receive equal contributions from ψ on this site. In more complicated injection schemes each ψ_x could influence a wider neighborhood on the fine lattice with more smoothly decaying strength. In $D > 1$ dimensions one has to choose some hierarchy of lattices similarly as in blockspin renormalization schemes, but otherwise no essentially new ingredients come in.

As discussed in [2,12,13] it is most efficient to implement MGMC recursively if this is practical. Then, for $l > 0$, an MGMC-update on Λ_l is defined in terms of an MGMC-update on Λ_{l-1} (and conventional steps), and only for $l = 0$ the definition becomes explicit. Schematically, this looks as follows:

```
procedure MGMC(l, φ, S_l)
sweep(φ, S_l, Λ_l)
  if l > 0 then
  construct S_{l-1}(.)
  ψ ← 0
  for j = 1 until γ do MGMC(l − 1, ψ, S_{l-1})
  φ ← φ + p_{l,l-1} ψ
  sweep(φ, S_l, Λ_l)
  endif
end
```

Here the procedure sweep is some conventional local update. Applied on the various coarse grids, it composes MGMC. Popular values for γ are $\gamma = 1$ (V-cycle) and $\gamma = 2$ (W-cycle). Since each consecutive coarser lattice has fewer sites than the previous one, a work estimate for a basic L^D lattice,

$$\begin{aligned} \text{work} &\propto L^D + \gamma(L/2)^D + \gamma^2(L/4)^D + \dots \\ &= L^D \frac{1}{1 - \gamma 2^{-D}} \end{aligned} \tag{19}$$

is expected to hold for one execution of MGMC(b, φ, S) on Λ_b in the recursive realization. It is thus only a constant factor more expensive than a simple local sweep as long as $\gamma < 2^D$. For the free field case and $\gamma = 2$ it can be proven [12,13] that MGMC leads to $z = 0$. For $\lambda\varphi^4$-theory at some moderate coupling strength it has turned out, however, that the improvement over local heatbath is most likely only a constant factor with no reduced exponent. The gain decreases as one approaches criticality at stronger and stronger fixed nonlinearity λ. In [13] a plausible explanation is given for this failure: In the nonlinear critical case the bare potential is of the double well shape, and the φ_x typically are sitting in one of the two wells. MGMC with piecewise constant injection can also be viewed as a hierarchy of constant shifts of the fields on cubic blocks of the basic lattice. On larger blocks such moves will almost always tend to move some φ's out of the wells. This will lead to only very small shifts being accepted. Collective rearrangements of fields from one well to the other, which are expected to be adequate in the Ising limit, do not occur in this way. In this picture one thus expects that in a more and more nonlinear situation fewer and fewer coarse lattices lead to efficient moves.

Mack and Meyer [14] proposed a version of MGMC based on a more complicated injection operator motivated by renormalization group ideas. It leads to a more complicated set of S_l and cannot easily be implemented recursively. Therefore their work estimate, analogous to (19), is $L^D \log L$ for $\gamma = 1$. First results on 12^4 indicate an overall gain, but more detailed analyses for the trend—presumably in lower dimensions—have to be awaited for. A nice byproduct of the approach [14] is information about the effective potential that follows from the multigrid effective action on the coarsest lattice.

MGMC, as other acceleration schemes, has additional problems with gauge invariant models. The basic problem is that the numerically slow or fast variation of gauge variant fields may not correspond to physical modes. In a two dimensional gaussian scalar model with background gauge fields some nice results were recently obtained [15]. The authors replace piecewise constant injection by a covariantly slowly varying shift on blocks. It is determined by the smallest eigenvalue of the covariant laplacian restricted to the block.

I would like to resume that MGMC is a versatile scheme, which may be adapted in some form to a large class of models. One then *does* perform some sort of global updates on all scales. Nevertheless, the dynamical success of MGMC is an open question to be decided on a case by case basis. The shape of the global moves, which are proposed by MGMC, may not be adequate. This will presumably always mean, that they are accepted under the constraint of detailed balance with an amplitude that is too small to reduce slowing down. So far, I do not know of a non-gaussian application, where a reduced dynamical exponent z has been demonstrated. It will be interesting to see if this can be done for asymptotically free spin models [13]. For some of them, cluster algorithms, which let the system determine the shape of global updates, seem to eliminate critical slowing down completely.

4. PERCOLATION CLUSTER ALGORITHMS

In a simple bond percolation model one activates the links on a cubic lattice independently with a certain probability. Then one analyzes the typical connectivity of sites, where an active bond is considered as conducting and the remaining ones as insulators. Above a critical activation probability, arbitrarily distant sites have a finite probability of being connected as parts of an infinite cluster that forms. In cluster algorithms, as pioneered by Swendsen and Wang (SW) [16], one builds similar clusters *correlated* with spin fields such that they can be used to propose dynamically appropriate collective moves of the spins.

The SW algorithm for the nearest neighbor Ising model is easily derived by the introduction of link variables in addition to the spins [17,18]. The partition

function can be transformed

$$Z = \sum_{\{\sigma_x = \pm 1\}} \exp\left(\beta \sum_{<xy>} (\sigma_x \sigma_y - 1)\right)$$

$$= \sum_{\{\sigma_x\}} \prod_{<xy>} [(1-p) + p\delta_{\sigma_x,\sigma_y}]$$

$$= (1-p)^{\#links} \sum_{\{\sigma_x, k_{xy}=0,1\}} \left(\frac{p}{1-p}\right)^{\sum k_{xy}} \Delta(k|\sigma) \tag{20}$$

Here k_{xy} are bond variables with values 0 (passive) and 1 (active), and the bond probability is given by

$$p = 1 - e^{-2\beta}. \tag{21}$$

The constraint

$$\Delta(k|\sigma) = \prod_{<xy>} (\delta_{k_{xy},0} + \delta_{k_{xy},1}\delta_{\sigma_x,\sigma_y}). \tag{22}$$

allows active bonds only between parallel spins and forces the spins on each cluster to be all parallel. By construction, the summation over k_{xy} in **(20)** restores the correct action for the pair of spins on each bond. Spin correlations evaluated in the σ- or in the (σ, k)-formulation therefore coincide. Summing over the spins in **(20)**, one finds the Fortuin Kasteleyn representation in terms of bonds only,

$$Z = \sum_{\{k_{xy}\}} \left(\frac{p}{1-p}\right)^{\sum k_{xy}} 2^{\nu[k_{xy}]}. \tag{23}$$

where $\nu[k_{xy}]$ is the number of clusters for bond configuration k_{xy}, and an overall factor has been omitted. SW's smart idea is now to keep both fields and update them in alternate order by the steps:

- for fixed σ_x activate k_{xy} independently link by link with probability $p \times \delta_{\sigma_x,\sigma_y}$
- for fixed k_{xy} assign a random spin value ± 1 to each cluster as a whole.

It is clear that under this procedure large clusters of spins can flip collectively, and that is has a chance of improving critical slowing down. A crucial ingredient is an algorithm that identifies the clusters of a given k_{xy} configuration in o(volume) operations. Such algorithms are known from percolation, and a rather efficient one is described in [19].

Once the cluster information is available from updating, it can be used further to measure reduced variance estimators [20] for spin observables $A(\sigma)$. The key formula is

$$\langle A(\sigma)\rangle = \langle \tilde{A}(k)\rangle = Z^{-1} \times$$

$$\sum_{\{k_{xy}\}} \left(\frac{p}{1-p}\right)^{\sum k_{xy}} \sum_{\{\sigma_x\}} \Delta(k|\sigma)\tilde{A}(k), \tag{24}$$

which becomes an identity upon insertion of

$$\tilde{A}(k) = 2^{-\nu[k_{xy}]} \sum_{\{\sigma\}} \Delta(k|\sigma) A(\sigma). \tag{25}$$

The cluster observable takes into account many configurations that could be chosen with equal probability, but of which only one is actually sampled in the computer. For Green functions like e. g. $A = \sigma_x\sigma_y$ the sum in **(25)** is easily carried out and gives

$$\tilde{A} = \sum_{i=1}^{\nu} \theta_{c_i}(x)\theta_{c_i}(y) =$$

$$= \begin{cases} 1 \text{ if x and y are in the same cluster} \\ 0 \text{ else} \end{cases} \tag{26}$$

with the characteristic function θ_c for cluster c. The advantage of $\tilde{A}$ for a small positive correlation at large separation $x - y$ can also be seen as follows: if $\langle\sigma_x\sigma_y\rangle = \epsilon$, then the variance of A (averaging over ± 1) is $1 - \epsilon^2$ while that of $\tilde{A}$ (averaging over $0, 1$) is $\epsilon(1-\epsilon)$. As a consequence, as one analyzes exponentially decaying correlations, the signal does not vanish in distance independent noise, but the noise itself decays exponentially. Of course, the rate is only half as fast as for the signal, and noise will eventually win out, but the practical improvement is quite useful. Under similar circumstances but with a different algorithm this has been demonstrated in [17]. It turned out, that the behavior derived for a fixed pair of spins here, is

also seen for time slice correlations averaged over the volume. Deeply in the broken phase, the cluster estimators are less helpful [21] as correlations do not decay to zero. For higher n-point functions we have no simple argument for the variances. In [22,23], however, cluster estimators proved to be crucial in determining mass gaps and renormalized 4-point couplings.

In [24] I proposed a variation of the SW-algorithm in the form of the single cluster (1C) method and at the same time its extension to $O(n)$ nonlinear σ-models. The 1C-idea is to build only one of the many SW-clusters as follows:

- pick a random site $x_0 \in \Lambda$
- build the cluster C connected to x_0 using the same bond probability as for SW
- flip all spins in C

This elementary update step can be executed by o($|C|$) operations and thus fluctuates in size. The important difference, as compared to SW, stems from the first step of 1C: If one imagines the whole lattice already decomposed into SW-clusters, then one of the c_i becomes C with a probabilitiy proportional to its weight $|C|$ (number of spins). One chanels relatively more CPU-work into large clusters and can thus expect a further reduction of autocorrelation times for long wavelength modes. Simple cluster estimators like (26) are easy to translate into the 1C scheme. The cluster sum in (26) is also done stochastically, and the extra probability $|C|/|\Lambda|$, with which 1C picks a cluster, is compensated,

$$\langle \sigma_x \sigma_y \rangle = \langle \frac{|\Lambda|}{|C|} \theta_C(x) \theta_C(y) \rangle^{1C}. \qquad (27)$$

As I sum both sides over x, y, I derive

$$\chi = \frac{1}{|\Lambda|} \langle (\sum_{x \in \Lambda} \sigma_x)^2 \rangle = \langle |C| \rangle^{1C}. \qquad (28)$$

In (26)–(28) it becomes obvious that the Ising critical point is strictly linked to the percolation threshold for the clusters involved. The geometrical size of the typical update step is automatically tied to the spin physics via the magnetic susceptibility in the simple relation (28).

In their original paper SW [16] quote for the Ising model

$$z \approx \begin{cases} 0.35 \text{ for } D = 2 \\ 0.75 \text{ for } D = 3 \end{cases} \qquad (29)$$

from monitoring and fitting the energy autocorrelation function at large time separation. They simulate the model *at criticality* on L^D lattices, and L plays the role of ξ in this case. Recently, for $D = 2$, a more extensive study [25] on lattices up to 512^2 concludes that their data are slightly more compatible with a logarithmic growth of τ with L, i. e. $z = 0$. Clearly, a small value of z and the logarithmic form look very similar over the lattice sizes that are accessible to simulations and presumably need yet larger lattices to be disentangled. While the distinction would be interesting for the dynamics of the process, either form shows the power of the SW-like algorithms for simulations to unravel *static* properties of spin models on what today are very large lattices. In [26] I compared SW with 1C in terms of integrated autocorrelation times for energy and magnetic suceptibility. It turns out, that 1C has smaller autocorrelation times and corresponding (effective?) exponents in all cases considered: up to $L = 256$ in $D = 2$ and $L = 64$ for $D = 3$. The difference is most notable in $D = 3$ and for $\tau_{int,\chi}$, where it amounts to almost an order of magnitude on 64^3. A further comparative study has been carried out in [27]. Exponential autocorrelation times are studied in $D = 2, 3, 4$. The authors find equal exponents z in $D = 2$ and advantages for 1C in the higher dimensions. In particular, for $D = 4$, their value is $z = -0.10(15)$ for 1C, while the SW-algorithm most probably has $z = 1$. In [23] the standard SW-algorithm made it possible to investigate subtle effects connected with vacuum tunneling in the 4 dimensional Ising limit of ϕ^4 theory.

Also in [24] I showed how to extend cluster methods to nonlinear $O(n)$ invariant σ-models. They are of considerably more interest to field theorists than Ising-like models as they comprise asymptotically free models in two dimensions and Higgs fields with continuous symmetry breaking in four dimensions. There

have been earlier attempts to generalize SW to these models [28,17,18] and [29]. They were all based on the idea of applying general $O(n)$ rotations to suitably constructed clusters. In rather extended simulations with an independently developed algorithm along the lines of [29] I had been unable to verify reduced values for z on large two dimensional lattices, and a similar experience was made in $D = 4$ in [30]. In contrast, the algorithm [24] may be understood as stochastically embedding [31] *Ising* variables into the $O(n)$ model defined by the partition function

$$Z = \int_{S_{n-1}} \prod_x d\mu(s_x) \exp\left(\beta \sum_{<xy>} s_x \cdot s_y\right). \quad (30)$$

Here the unit vector spins s_x from the sphere S_{n-1} in n dimensional space are integrated with the $O(n)$ invariant measure $d\mu$. The embedding is with respect to a randomly chosen direction $r \in S_{n-1}$ by writing

$$s_x = s_x^{\perp} + r|s_x \cdot r|\sigma_x \quad (31)$$

with the Ising field σ_x giving the sign of the r-components of s_x. This Ising model is updated with the effective action

$$S_{eff} = -\beta \sum_{<xy>} |s_x \cdot r \; s_y \cdot r|\sigma_x\sigma_y. \quad (32)$$

following form (31). The SW-algorithm may be trivially adapted to this ferromagnetic system with bond dependent interaction strength. While in [31] a simulation is carried out precisely in this fashion, in [24] I immediately described the 1C-version of the embedded algorithm. There the elementary update step consists of the following:

- pick a random $r \in S_{n-1}$ and a random site x_0
- grow one cluster C starting from x_0 with bond probability $(1-\exp(-2\beta s_x \cdot r \; s_y \cdot r))\theta(s_x \cdot r \; s_y \cdot r)$
- reflect for all $x \in C : s_x \to s_x - 2(s_x \cdot r)r$

The extension of **(27)** reads in this case

$$\langle s_x \cdot s_y\rangle = n\langle \frac{|\Lambda|}{|C|}\theta_C(x)\theta_C(y) \; s_x \cdot r \; s_y \cdot r\rangle^{1C}. \quad (33)$$

The form of the bond probability guarantees that I average in **(33)** over positive quantities only. As a consequence, variance reduction and the connection between the spin model criticality and the percolation threshold of the clusters involved is just as in the Ising model. It may be worthwhile to contrast the principal strategy of the new 1C update algorithm with standard methods: There, a certain local move is proposed and either accepted or rejected. Resonable acceptance is achieved by proposing small enough moves. With the heatbath, this is done—so to speak— automatically on the average. In the new algorithm, I pick a transformation and apply it locally somewhere. If this turns out to be a large move for the local spin direction inflicting a large energy penalty, then I extend the move to a growing domain of spins. This stops when a surface of spins is reached where our move becomes energetically acceptable. It is obviously important here, that my moves belong to the gobal symmetry group, as this is the reason that the energy does not change in the interior of clusters. Also the boundary conditions have to be symmetric, as was always implicitly assumed here (e. g. periodic).

The cluster algorithms desribed above find a growing number of applications these days [30–40]. My own first test was in the two dimensional XY-model [32]. The 1C algorithm has extremely short autocorrelation times both in the vortex and in the spinwave phase as shown in Figs.1,2. The most amazing result is $\tau_{int,\chi} \approx 0.14$ in the vortex phase for all β where the correlation length is small compared to the lattice size. This study included correlation lengths up to 69 which can now be reached very easily in two dimensional systems. The τ-values smaller than one are due to the fact that correlation times between estimates separated by a 1C step have been converted to the unit "flip per spin" so that it is comparable with "sweeps" in CPU time. As I compare these numbers with results [31] employing the embedded SW algorithm, I find that $\tau_{int,\chi}$ is typically a factor $\simeq 10$ smaller for the 1C method

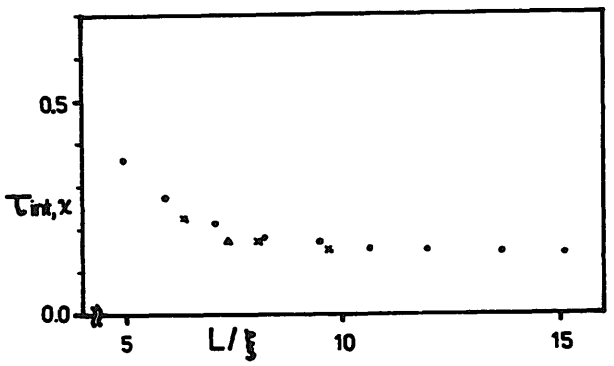

FIGURE 1

Autocorrelation times $\tau_{int,\chi}$ in units comparable to sweeps for the two dimensional XY-model in the vortex phase at correlation length ξ. Lattice sizes are $L = 128$ (•), $L = 256$ (×) and $L = 512$ (△).

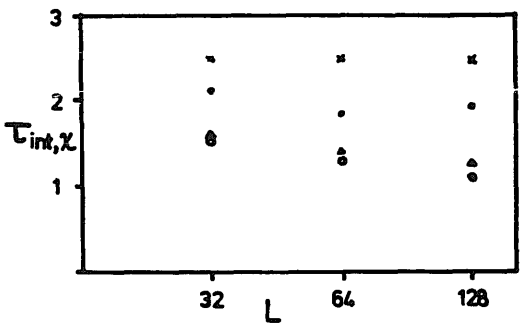

FIGURE 2

Similar to Fig. 1, but $\beta = 1.07$ (o), and, in the spinwave phase, $\beta = 1.12$ (△), 1.30 (•), 1.50 (×).

with the same parameters in the scaling regime of the vortex phase. Both results are suggestive of the complete absence of critical slowing down. The XY-model physics was analyzed in quite some detail in [32]. I interpret my results as supporting the Kosterlitz Thouless picture.

Next, I conducted a similar study for the $O(3)$-model which is expected to be asymptotically free and therefore resembles QCD. Again no sign of critical slowing down was visible as I went to correlation lengths of up to 122 on a 512 × 800 lattice. A brief resume of the physics seen is: there are *still* problems with asymptotic scaling with the rather enhanced accuracy possible now, but good evidence is seen for scaling in relations between physical renormalized quantities. For more details both on the physics as well as the algorithm I refer the reader to [33,34].

In [38] an interesting application in four dimensions has been made to the $O(4)$ model as a simplified setting to study Higgs physics. Here, as for the one component model [22], variance reduction was most important at moderate correlation length in the symmetric phase. In the broken phase, it has been shown [30] that a cluster algorithm is crucial to eliminate critical slowing down from the presence of Goldstone bosons.

It is obvious, that all the successes with the percolation method in $O(n)$ spin models call for extensions to other systems and in particular to gauge theories. Some very important progress has been made in this direction in [41] and independently with a somewhat less efficient method in [42]. In both cases $Z(2)$ gauge theory in $D = 3$ is simulated at the critical point dual to the Ising spin system. I consider [41] as an Ising gauge theory counterpart of the SW-algorithm for spin systems. For details on [41,42] I refer to contributions to these procedings by P. Lauwers and R. Brower. At present it is not yet clear how to progress toward $U(1)$ gauge theory. The nonabelian case looks even harder, as the transition to plaquette variables in [41] is not possible there. Nevertheless, generalizing the experience with $O(n)$-models, the goal of extending cluster methods to gauge theory seems so desirable, that the matter clearly deserves a lot more thought.

Finally I would like to mention [43], where cluster and multigrid methods have been amalgated by forming multigrids of a shape determined stochastically by the system. One is then able to exert some control over the size distribution of the moves occuring. In view of the extraordinary success of the plain cluster algorithms for spin systems, I have some doubt about the competitiveness of these more complicated schemes there. Subjects of ongoing research include

cluster methods for RP_{n-1}, CP_{n-1} and $SU(n) \times SU(n)$ nonlinear σ-models and spin glasses. They are interesting by themselves, and their treatment may well be an important stepping stone toward gauge theory.

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